Vertex-minor universal graphs for generating entangled quantum subsystems

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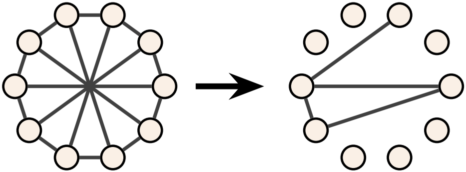
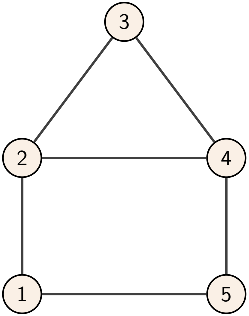


Basic definitions

Graphs

Definition (Graph)

A graph $G = (V, E)$ is composed of a set of vertices V and a set of edges E. Here, the graphs are undirected (no directed edge) and simple (no self-loop and at most one edge per pair of vertices).

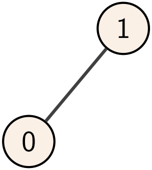


Graph states

Definition (Graph state)

Given a graph $G = (V, E)$, the corresponding graph state $|G\rangle$ is the quantum state

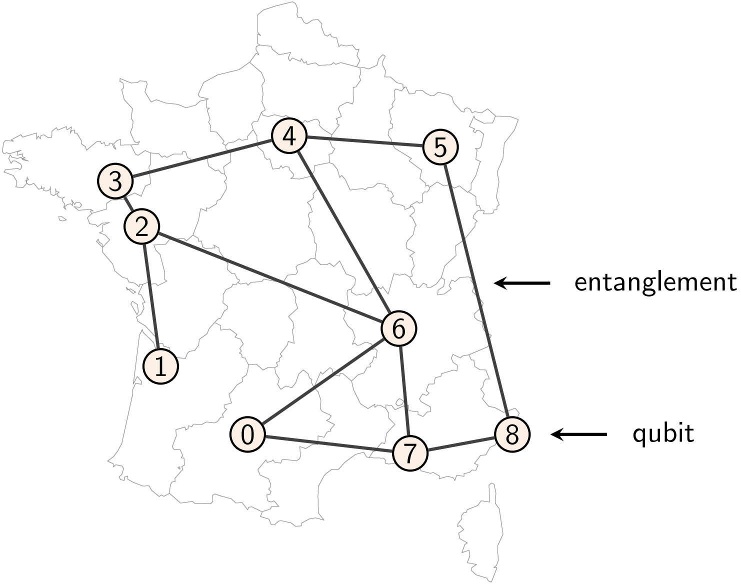
$$|G\rangle = \left(\prod_{(u,v)\in E} CZ_{u,v} \right) |+\rangle_V$$



$$\begin{aligned} |G\rangle &= CZ_{0,1} \left(|+\rangle_0 \otimes |+\rangle_1 \right) \\ &= \frac{1}{2} \left(|00\rangle + |01\rangle + |10\rangle - |11\rangle \right) \end{aligned}$$

The power of local operations on graph states

Quantum communication networks



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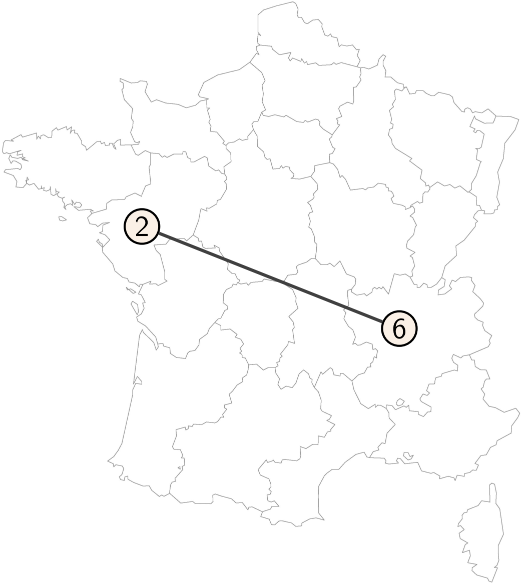

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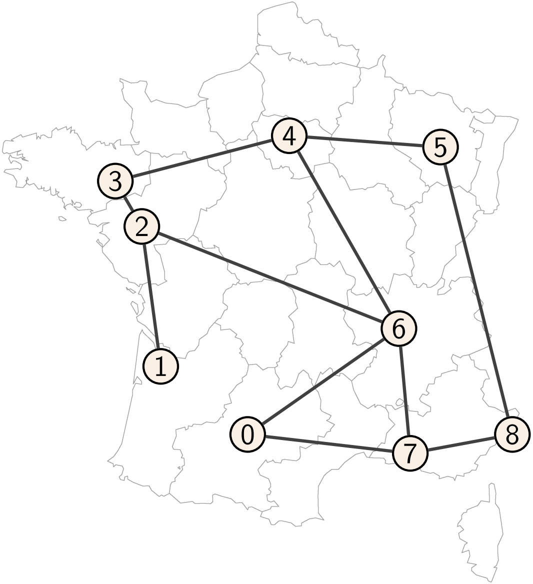

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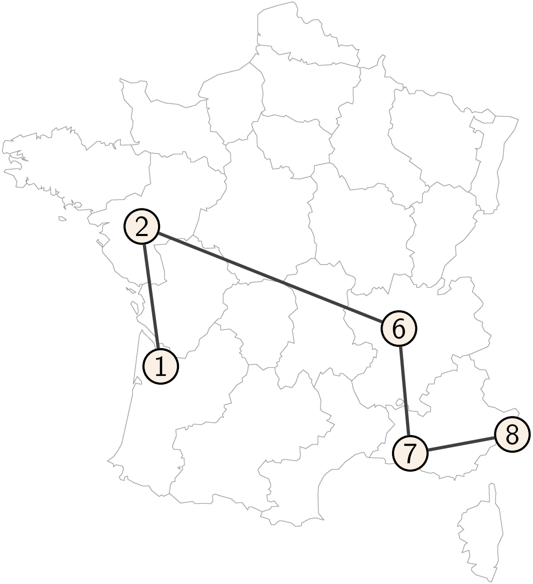

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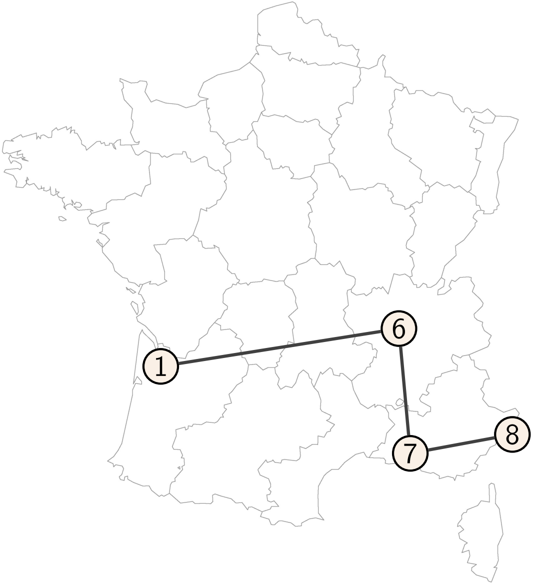

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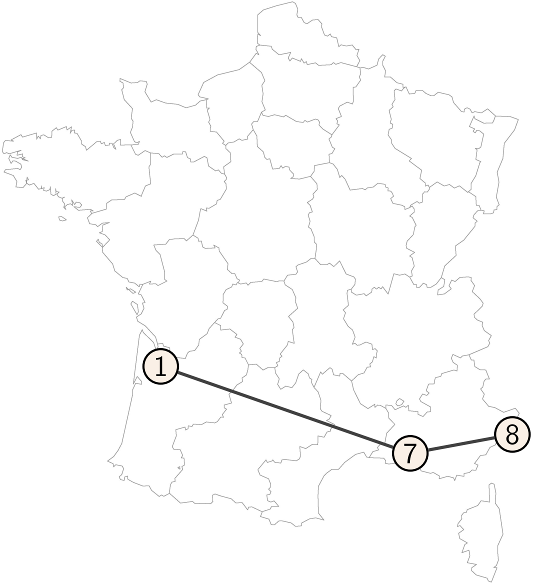

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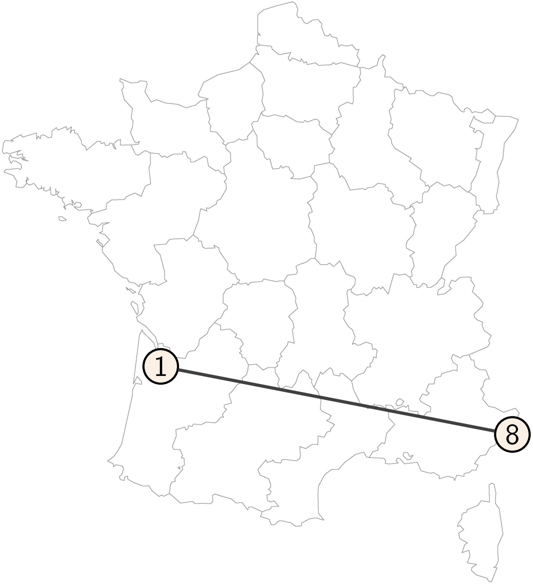

Problem: Generating arbitrary graph states

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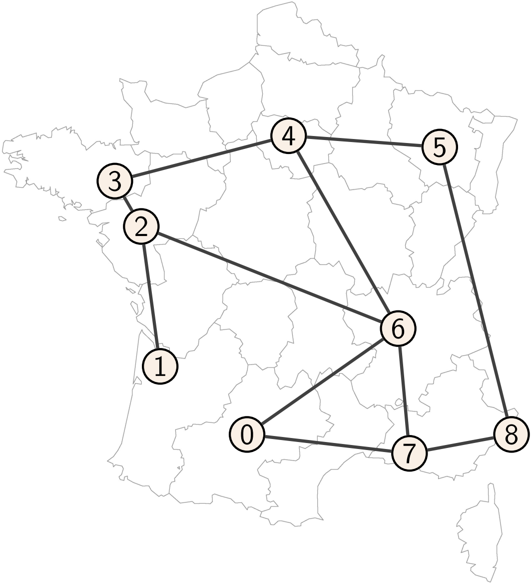
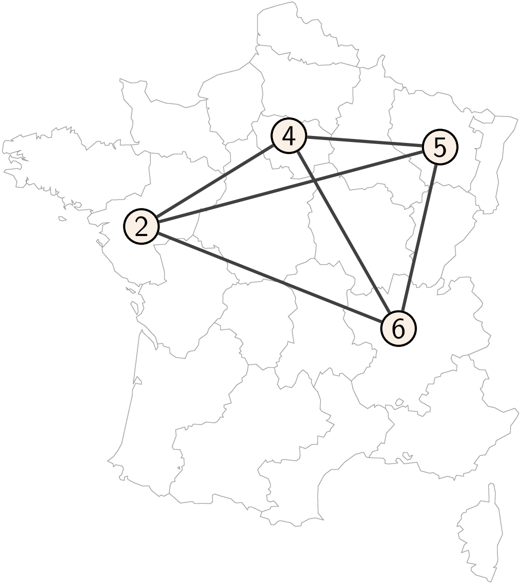
Natural question: What if we want to create any arbitrary **graph state** between any nodes?

Creating graph states



Creating graph states



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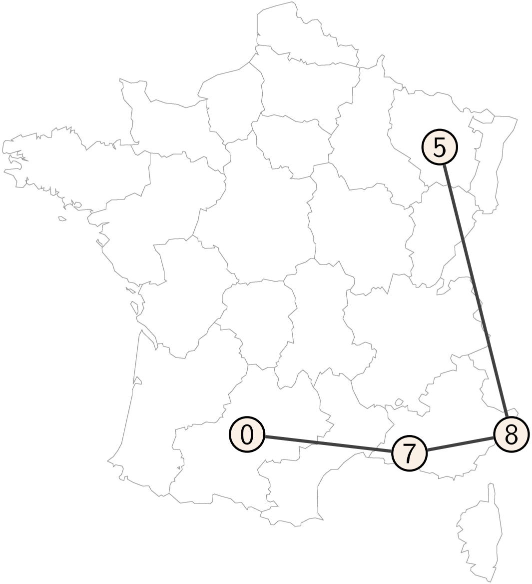

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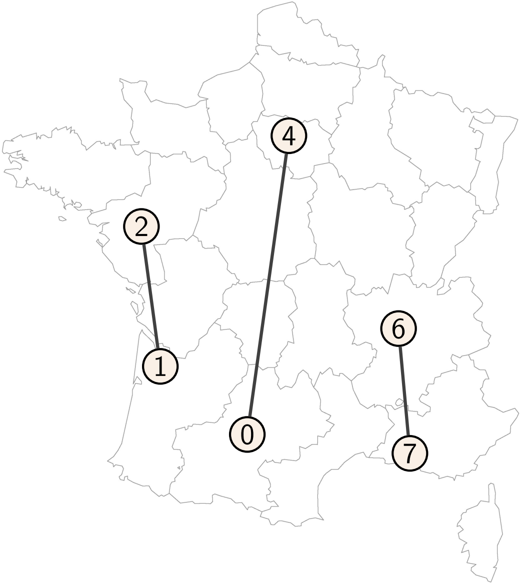
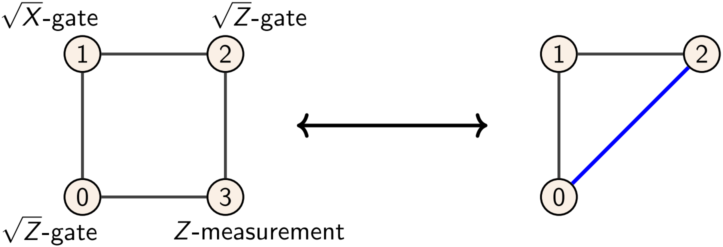


A graphical counterpart for local operations on graph states

Correspondence between graph states and graphs

$|G\rangle$ G



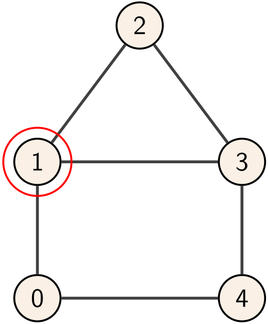
local (i.e. single-qubit) quantum operations *

vertex deletions & local complementations

Local complementation

Definition

A local complementation on a vertex u consists in complementing the (open) neighborhood of u.



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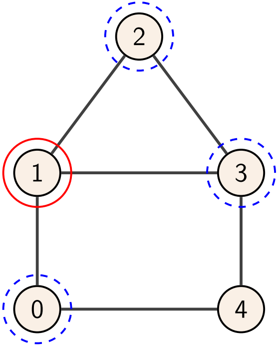

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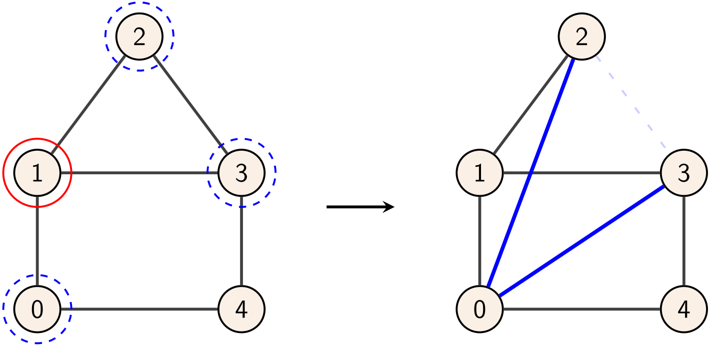
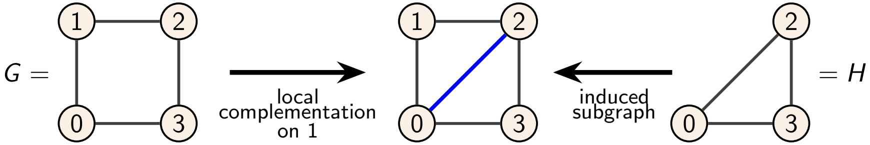


Vertex-minors

Definition (Vertex-minor)

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ such that $V_H \subseteq V_G$, H is a vertex-minor of G if H can be obtained as a induced subgraph of G by means of local complementations.



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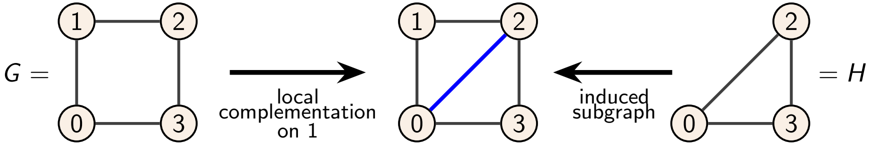
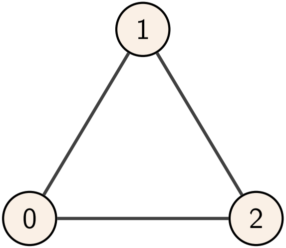


Definition

A graph G is *k-vertex-minor universal* if any graph on any k vertices is a vertex-minor of G.

k-vertex-minor universal graphs : example 1

K_3 is 2-vertex-minor universal.



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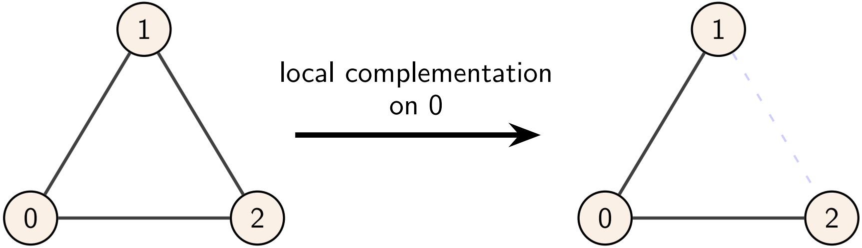

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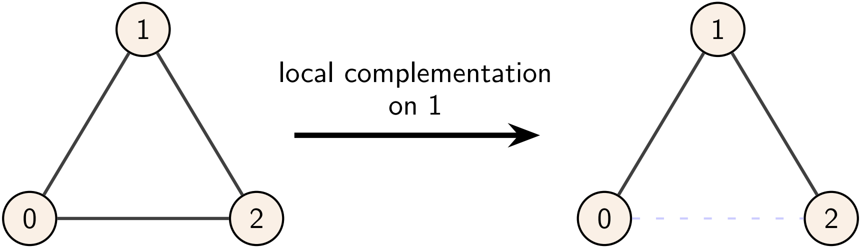

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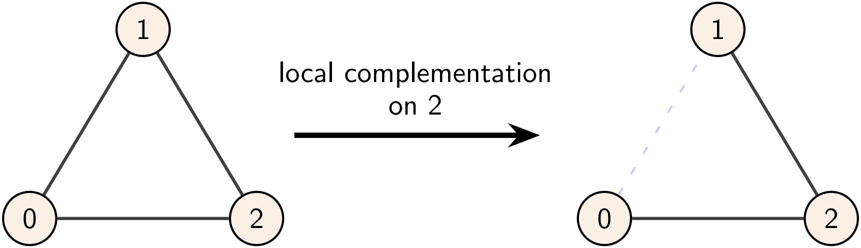
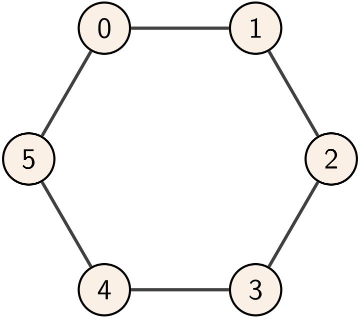


k-vertex-minor universal graphs : example 2

C_6 is 3-vertex-minor universal.



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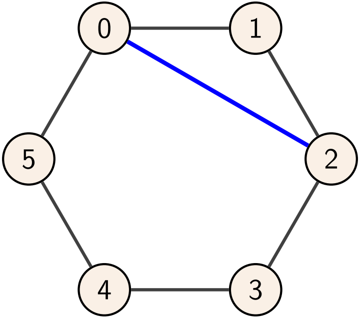

To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.

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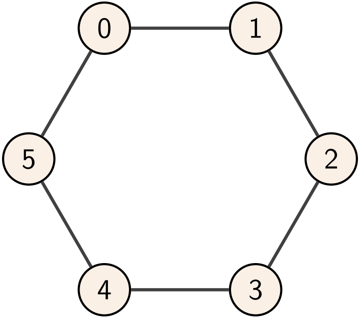

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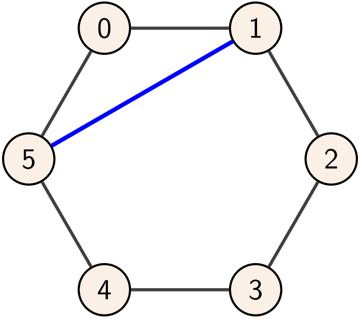

To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
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k-vertex-minor universal graphs : example 2

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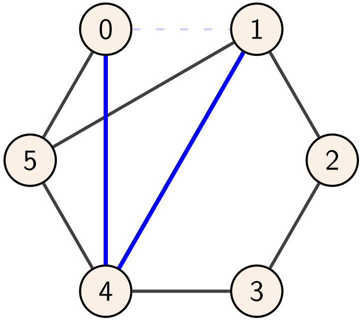

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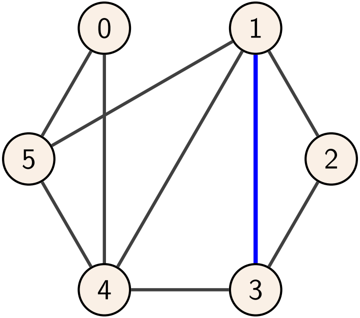

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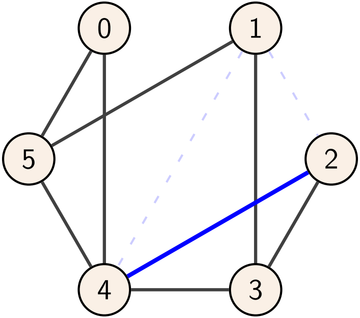

To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
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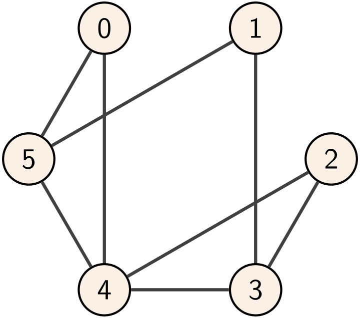


To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
To induce the empty graph on $\{0, 1, 2\}$: Local complementation on 0, on 5, on 2, on 3.

Quantum counterpart

Proposition

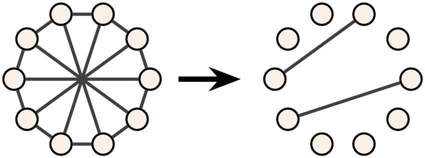
If G is k-vertex-minor universal, any graph state on any k qubits of $|G\rangle$ can be induced by local operations and classical communication.

Related work: pairability

Vertex-minor universality generalizes **pairability**, a notion introduced by Sergey Bravyi, Yash Sharma, Mario Szegedy, Ronald de Wolf in "Generating k EPR-pairs from an n-party resource state" (2022).

Definition

A quantum state is said *k*-pairable if any *k* EPR-pairs on any 2k qubits can be induced can be induced by local operations and classical communication.



For an arbitrary k, existence of k-vertex-minor universal graphs ?

An upper bound on k-vertex-minor universality

For an arbitrary k, existence of k-vertex-minor universal graphs ? Of reasonable size ?

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A lower bound:

Proposition

A k-vertex-minor universal graph is of order $\Omega(k^2)$.

An upper bound on k-vertex-minor universality

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A lower bound:

Proposition

A k-vertex-minor universal graph is of order $\Omega(k^2)$.

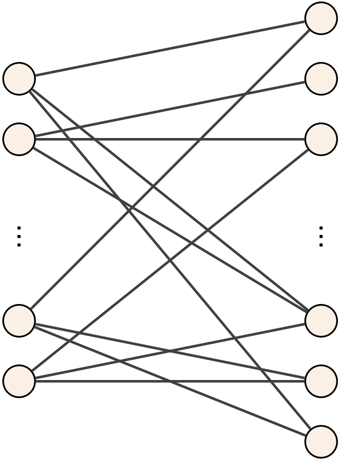
Proof.

- Given a fixed set of k vertices, there are at most 3^{n-k} vertex-minors up to local complementation.
- There are $\Omega(2^{k^2})$ different graphs of order k up to local complementation.

Random construction of k-vertex-minor universal graphs of order $\Theta(k^2)$

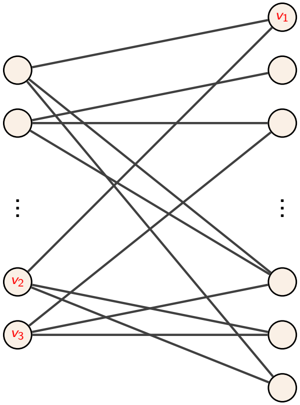
Outline of the construction

Random bipartite graph $G = (L \cup R, E)$ (the probability of an edge existing between L and R is $1/2$). $|L| = \Theta(k \ln(k))$, $|R| = \Theta(k^2)$.



Proof of vertex-minor universality: greedy algorithm

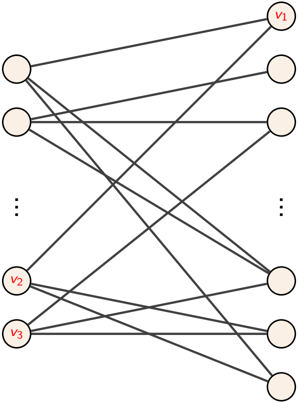
Given any fixed set of k vertices:



Proof of vertex-minor universality: greedy algorithm

Given any fixed set of k vertices:

- 1 - Move every vertex to the left by means of pivoting.



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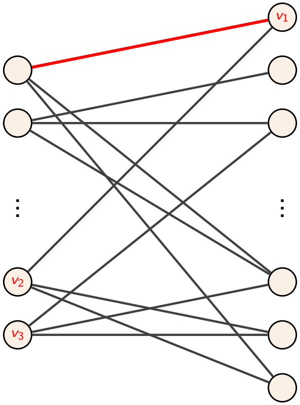

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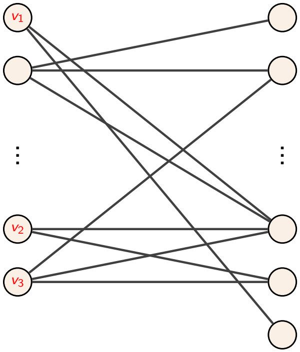

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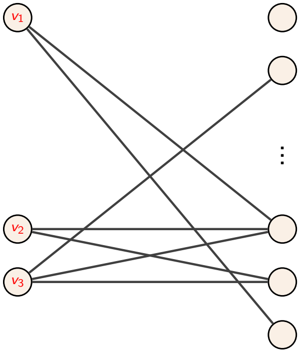
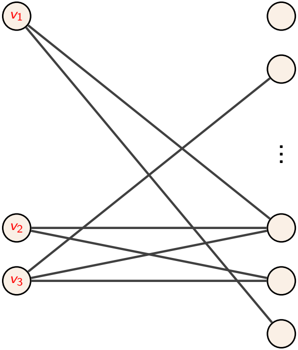


Proof of vertex-minor universality: greedy algorithm

Given any fixed set of k vertices:

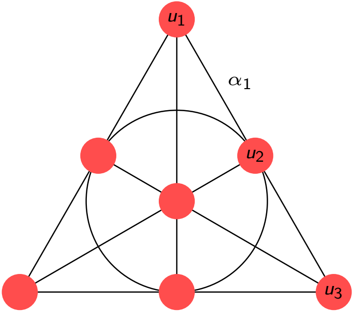
- 1 - Move every vertex to the left by means of pivoting.
- 2 - Check if the incidence matrix if of full rank.



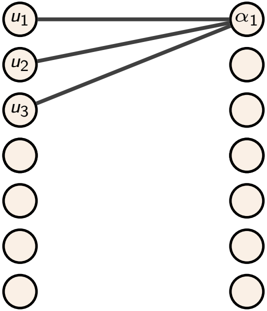
$$\begin{matrix} (v_1, v_2) & (v_1, v_3) & (v_2, v_3) \end{matrix}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Also: an explicit construction of order $\Theta(k^4)$

There is an explicit construction of k-vertex-minor universal graphs of order $\Theta(k^4)$ based on projective planes.



PG$(2, q)$: projective plane over $\mathbb{F}_q$



G_q: bipartite incidence graph graph of PG$(2, q)$

Summary

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k-vertex-minor universal graphs:

- Probabilistic construction of order $\Theta(k^2)$.
- Explicit construction of order $O(k^4)$.

Future directions:

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- What if we allow more than 1 qubit per party ?

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- What if we allow more than 1 qubit per party ?
- What about noise ?

Thanks

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