# Covering a Graph with Minimal Local Sets

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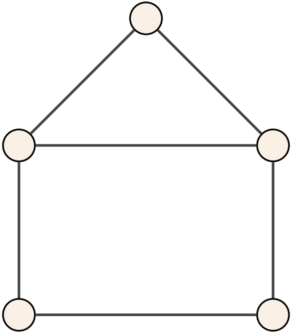
# Outline

1. Motivation : Quantum graph states
2. (Minimal) local sets
3. Links with the cut-rank function
4. Conclusion

# Disclaimer

This work only covers undirected[1] and simple[2] graphs.

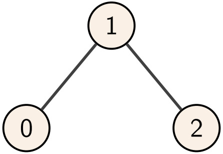


[1] Edges do not have a direction.
[2] No multiples edges and no loops.

# Motivation : Quantum graph states

# Quantum graph states

A graph state is a quantum state represented by an undirected and simple graph. The vertices represent the qubits[3] and the edges represent entanglement [4].




$$|G\rangle = \frac{1}{\sqrt{8}} \left( |000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle \right)$$

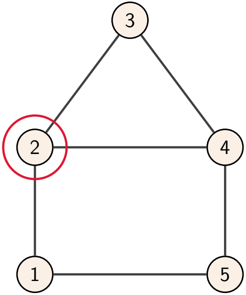
[3] The qubit is the quantum version of the classical binary bit.

[4] Two particles are entangled if they cannot be described independently of the state of the others

# Local complementation

**Definition (Local complementation)**

Given a graph $G$, a **local complementation** on a vertex $u$ consists in complementing the (open) neighborhood of $u$ in $G$.



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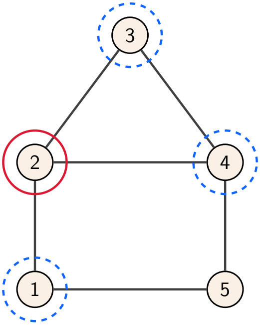

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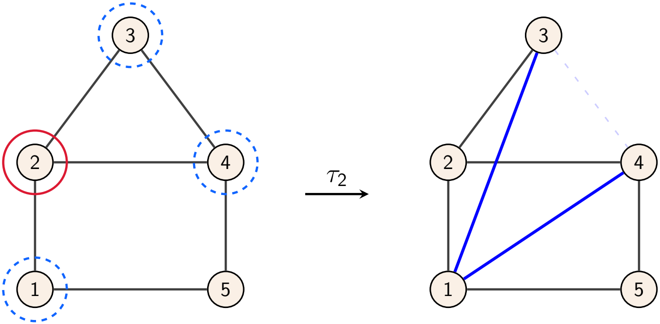
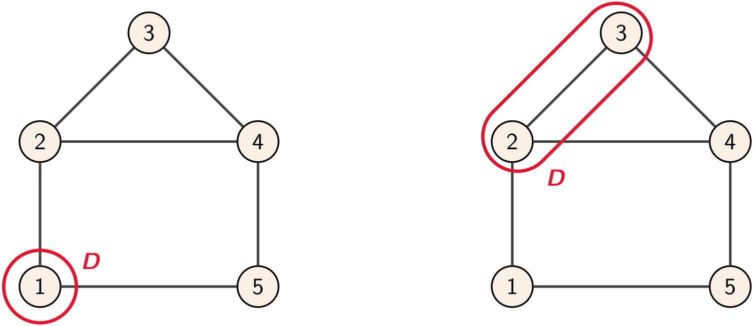


# (Minimal) local sets

# Odd neighborhood

**Definition (Odd neighborhood)**

Given a set of vertices $D$, the **odd neighborhood** $Odd_G(D)$ of $D$ is the set of vertices that are neighbors of an odd number of vertices in $D$.



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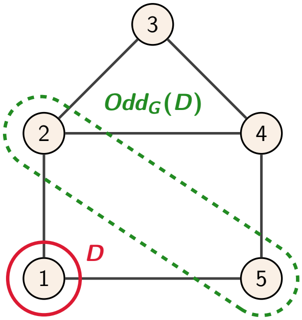


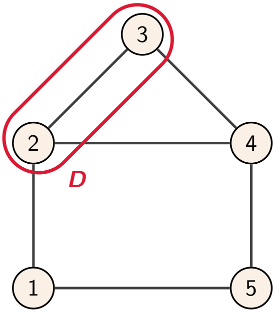

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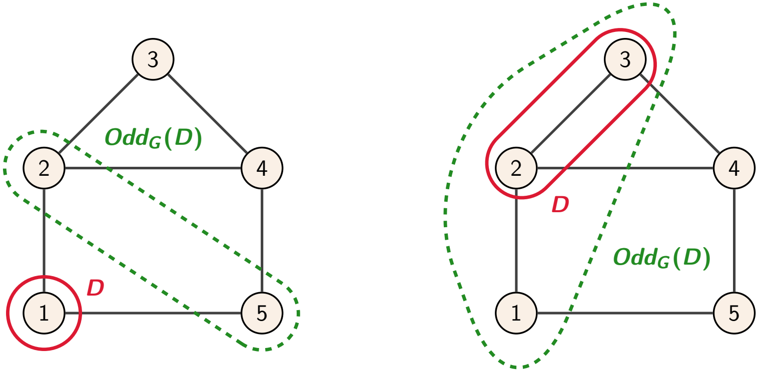
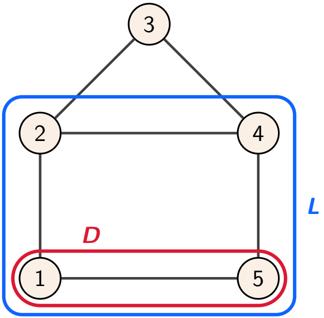


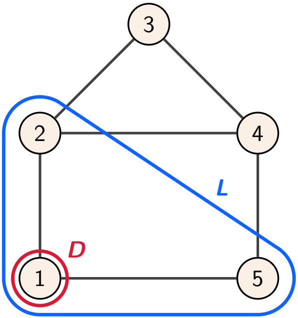
# (Minimal) local sets

**Definition**

A **local set** is a non-empty vertex set of the form $L = D \cup Odd_G(D)$.
A **minimal local set** is a local set that is minimal by inclusion (i.e it doesn't strictly contain another local set).




a local set

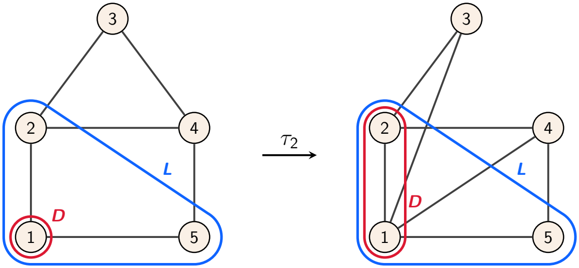



a minimal local set

# Properties of minimal local sets

**Proposition**

*(Minimal) local sets are invariant by local complementation.*



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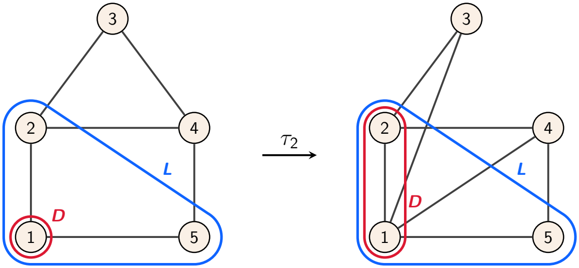


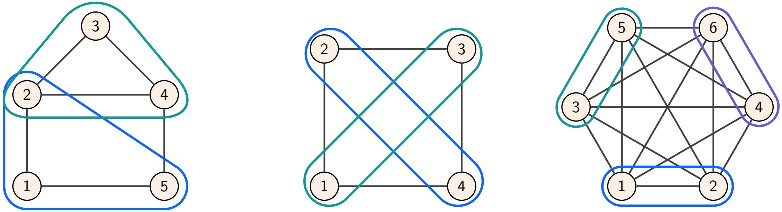

**Proposition (Høyer, Mhalla, Perdrix 2006)**

*Given a minimal local set $L$, for any $x \in L$, there exists a sequence of local complementations mapping $G$ to a graph $G'$, such that $L = \{x\} \cup N_{G'}(x)$.*

# Main result

**Theorem**

*Any graph is covered by its minimal local sets, i.e. every vertex is contained in at least one minimal local set.*



# Links with the cut-rank function

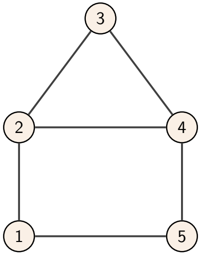
## The cut-rank function

**Definition (Cut-rank function)**

For $A \subseteq V$, let the cut-matrix $\Gamma_A = ((\Gamma_A)_{ab} : a \in A,\ b \in V \setminus A)$ be the matrix with coefficients in $\mathbb{F}_2$ such that $\Gamma_{ab} = 1$ if and only if $(a, b) \in E$. The **cut-rank function** of $G$ is

$$\begin{aligned} \text{cutrk} \colon 2^V &\longrightarrow \mathbb{N} \\ A &\longmapsto \textbf{rank}(\Gamma_A) \end{aligned}$$

$A \subseteq V$ is said **full cut-rank** if $\text{cutrk}(A) = |A|$.




$cutrk(\emptyset) = cutrk(\{1, 2, 3, 4, 5\}) = 0$

$cutrk(\{1, 5\}) = 2$ : full cut-rank

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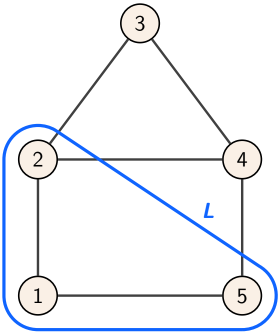
# Minimal local sets defined with the cut-rank function

**Proposition**

*Given a graph $G = (V, E)$ and $A \subseteq V$, $A$ is a minimal local set if and only if $A$ is not full cut-rank, but each of its proper subset is.*

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$cutrk(\{1,2\}) = cutrk(\{1,5\})$
$= cutrk(\{2,5\}) =$2: full cut-rank

$cutrk(\{1\}) = cutrk(\{2\})$
$= cutrk(\{5\}) = 1$: full-cut-rank

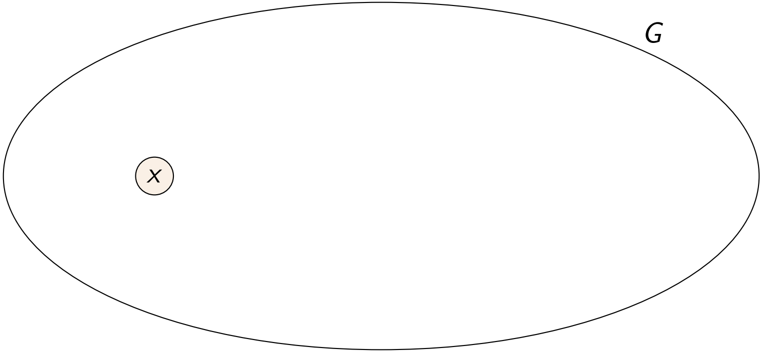
$cutrk(\emptyset) = 0$: full-cut-rank

## Algorithm

Given a vertex $x$, we want to find a minimal local set that contains $x$.

**Lemma**

*For any full-cut-rank set $A$, there exists a disjoint full-cut-rank set of same size.*



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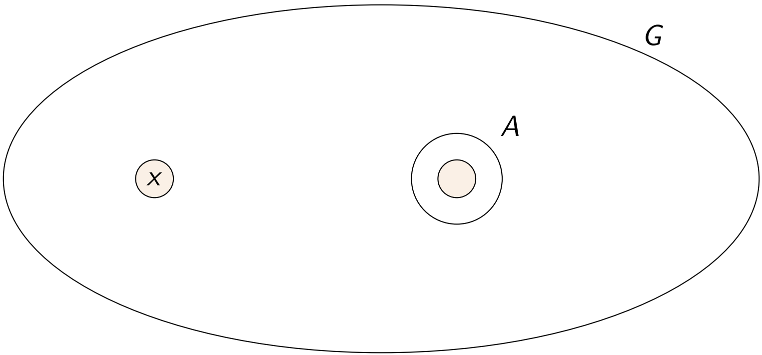

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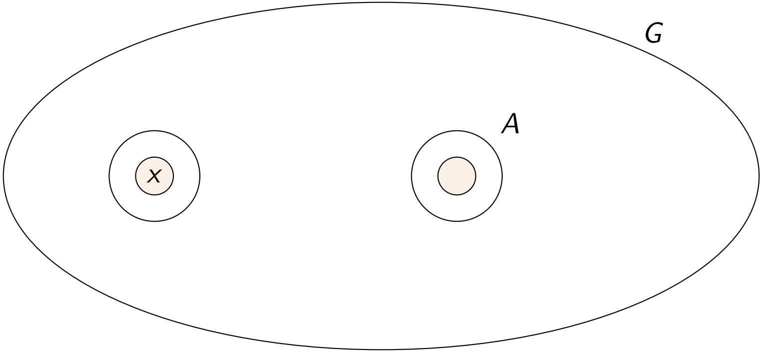

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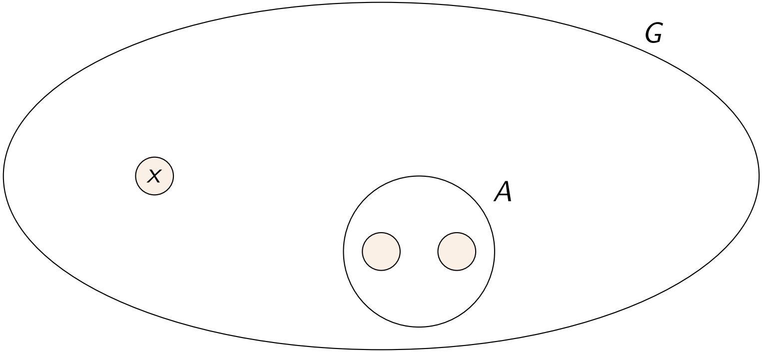

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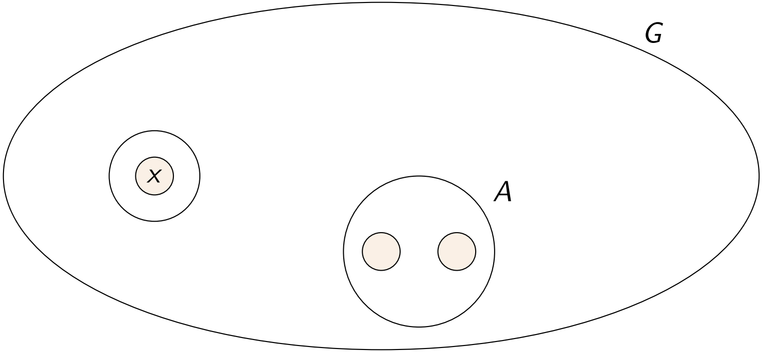

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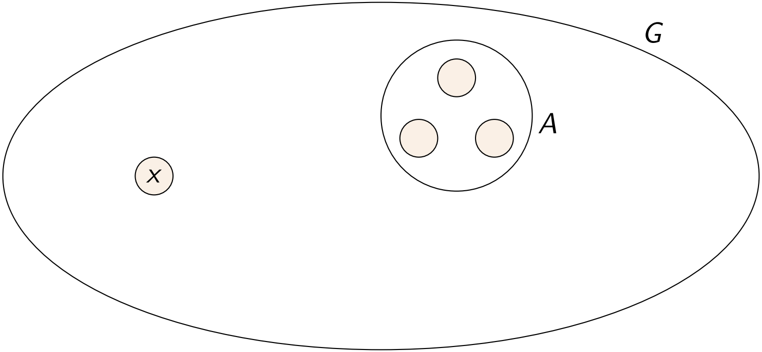

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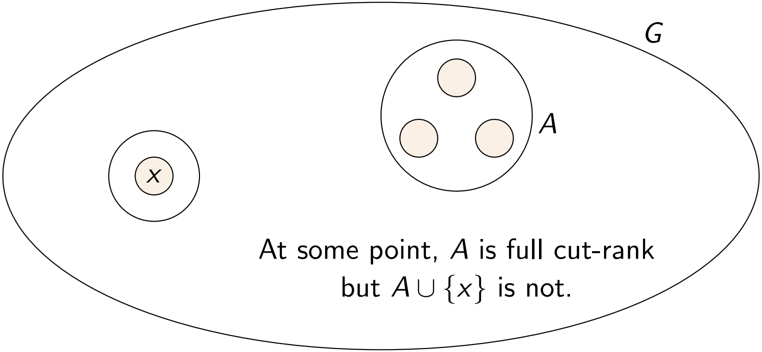

# Generalisation to any function with similar properties

The proof uses only the fact that the cut-rank function is a function with values in $\mathbb{N}$ which satisfies the following properties:

- **symmetry:** $\forall A \subseteq V,\ \mathrm{cutrk}(V \setminus A) = \mathrm{cutrk}(A)$,
- **linear boundedness:** $\forall A \subseteq V,\ \mathrm{cutrk}(A) \leqslant |A|$,
- **submodularity:**
  $\forall A, B \subseteq V,\ \mathrm{cutrk}(A \cup B) + \mathrm{cutrk}(A \cap B) \leqslant \mathrm{cutrk}(A) + \mathrm{cutrk}(B)$.

# Example of another use case: matroids

Let $\lambda_M$ be the connectivity function of a matroid $M$ with ground set $E$: if $X \subseteq E$, $\lambda_M(X) = r(X) + r(E - X) - r(M)$.

Define a $\lambda_M$-minimal local set as a set $A \subseteq X$ such that $\lambda_M(A) \leqslant |A| - 1$ but for every $B \subsetneq A$, $\lambda_M(B) = |B|$.

**Theorem**

*Any matroid $M$ is covered by its $\lambda_M$-minimal local sets, i.e. every element is contained in at least one $\lambda_M$-minimal local set.*

# Conclusion

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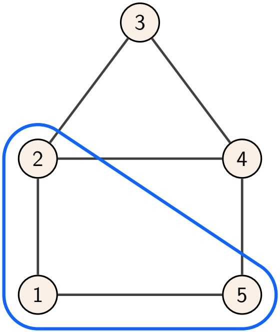
- Definition of minimal local sets using the cut-rank function.



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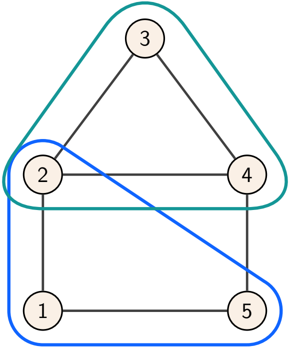
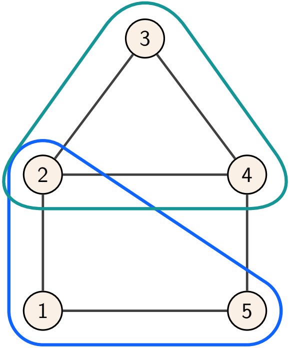
- Every vertex is contained in at least one minimal local set.



# Conclusion

- Definition of minimal local sets using the cut-rank function.
- Every vertex is contained in at least one minimal local set.
- The result generalises to any submodular, symmetric and linearly bounded function in $\mathbb{N}$ $\implies$ applications ?



# Thanks

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