Graph states and counter-examples to the LU-LC conjecture

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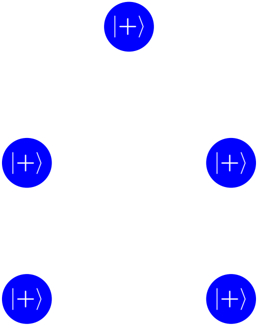
Quantum meets
16/09/26

universität innsbruck

Graph states

A graph state is a quantum state represented by an undirected[1] and simple[2] graph. The vertices represent the qubits and the edges represent entanglement.



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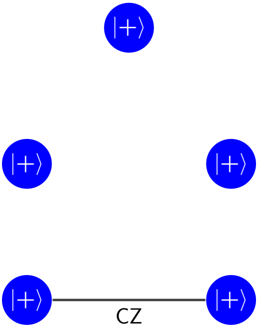

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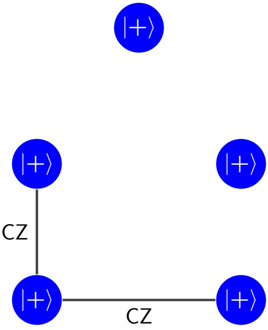

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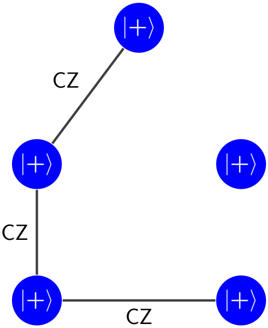

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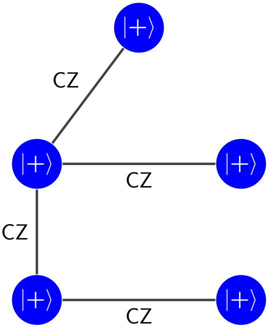

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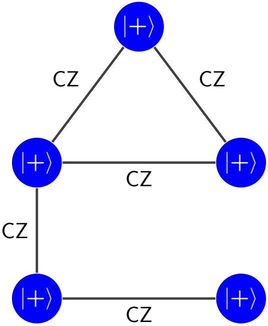

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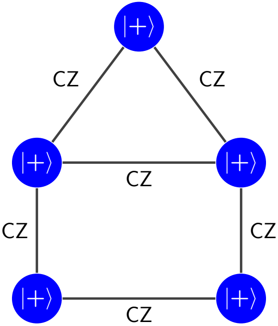

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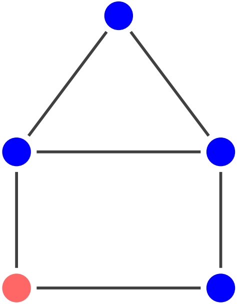
Stabilizer states

Graph states are a subfamilly of the stabilizer states.



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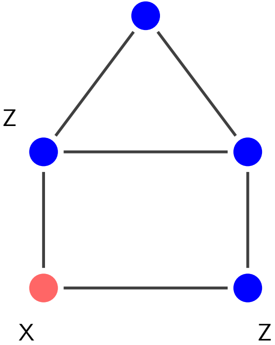

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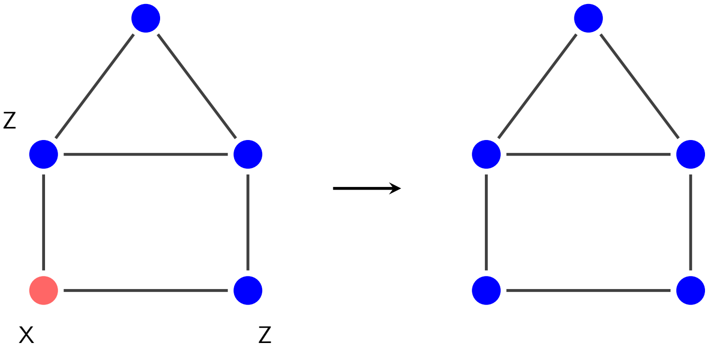
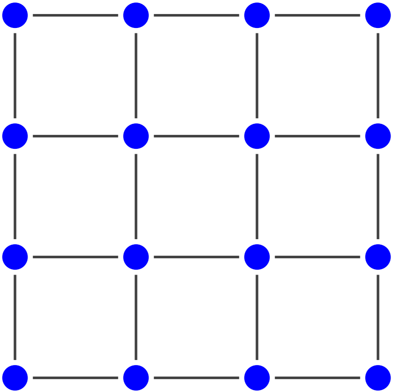


Measurement-based quantum computing (MBQC)

- Introduced in the early 2000's by Hans Briegel and Robert Raussendorf.
- Graph states are the resources for MBQC.



Briegel & Raussendorf, A one-way quantum computer, PRA 2001

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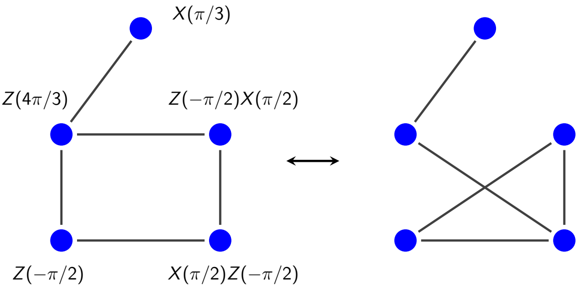
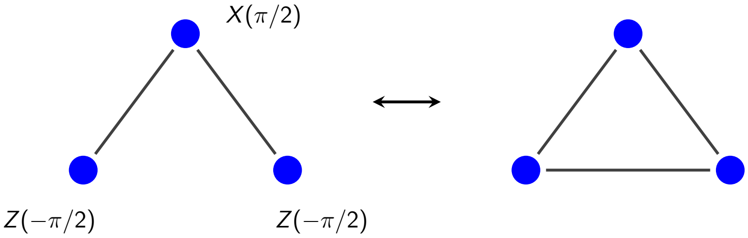


An easier subproblem: local Clifford equivalence

LC-equivalence (**L**ocal **C**lifford)
= related by local unitaries in the **Clifford** group: $H, X(\pi/2), Z(\pi/2)$...



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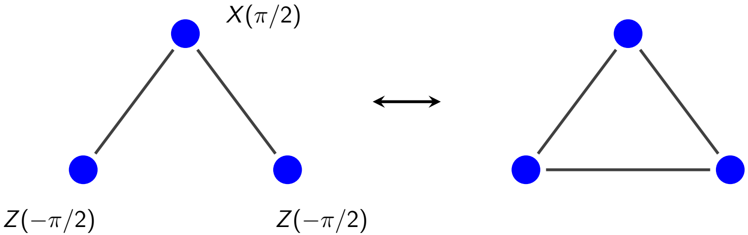


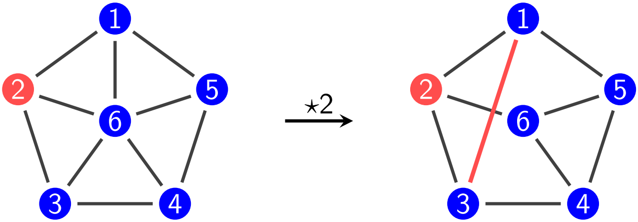
Theorem (Van den Nest, Dehaene, De Moor, 2004)

Two graph states are local Clifford equivalent if and only if the two corresponding graphs are related by ***local complementations****.*

Local complementation

Definition (Kotzig, 1966)

A local complementation on a vertex u consists in complementing the (open) neighbourhood of u.



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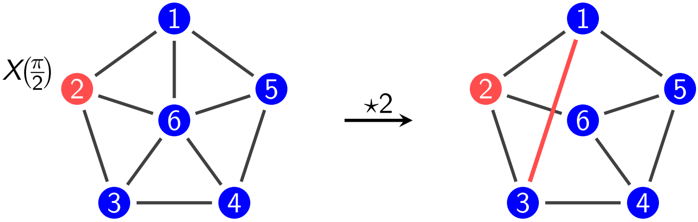
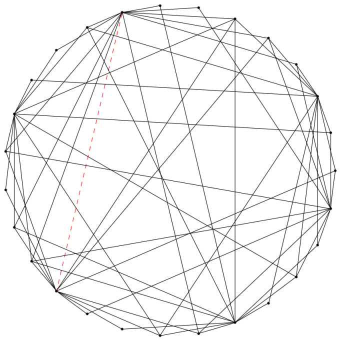


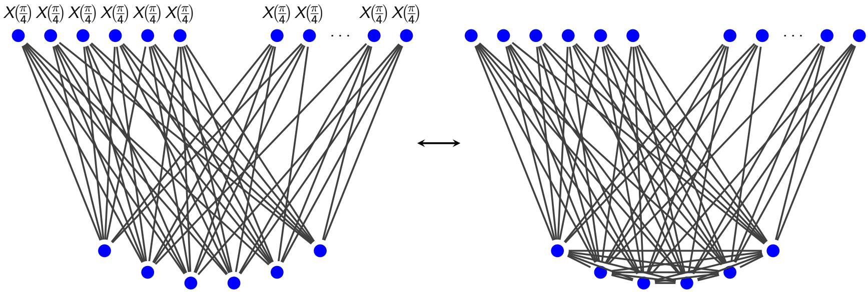
LU $\neq$ LC

The LU-LC conjecture (LU-equivalence and LC-equivalence coincide for graph states) was proven **false** in 2007 (Ji et al.).



(27 qubits)

Another look at the counter-example



Tsimakuridze & Gühne, Graph states and local unitary transformations beyond local Clifford operations, J. Phys. A 2017

Some results beyond the LU-LC conjecture

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  $\rightarrow$ include GHZ states, 1D cluster states, tree graph states, ring graph states, repeater graph states, ...

Minimality

Question: Is 27 qubits minimal ? (open since 2007)

Two cases:

- **Yes**:
- **No**:

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- **Yes**: LU=LC for graph states up to 26 qubits.
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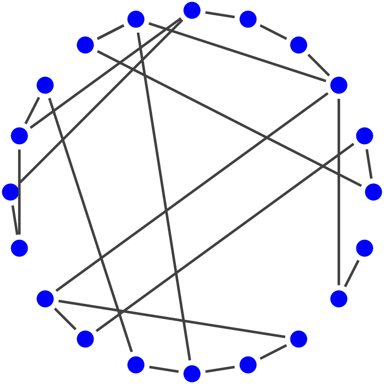
Answer : Yes!

Idea: test every graph state up to 26 qubits



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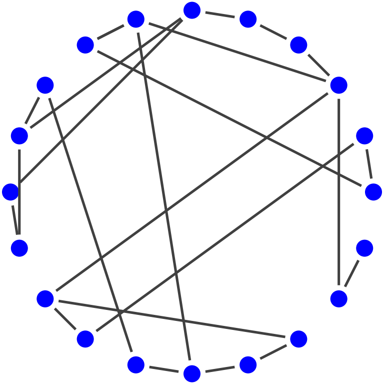

Problem: $> 2^{\binom{26}{2}} \sim 10^{98}$ graph states with up to 26 qubits.

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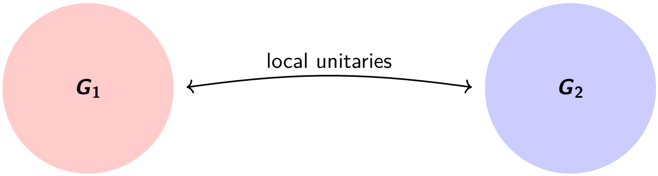
Theorem (C, Perdrix, 2025)

Two graph states are local unitary equivalent if and only if the two corresponding graphs are related by r***-local complementations*** *for some integer* r.

Proof sketch



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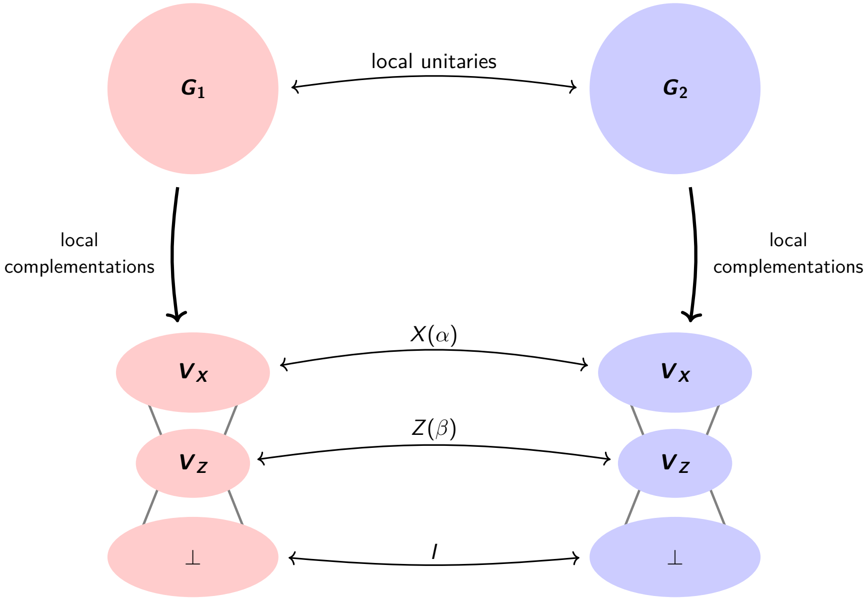

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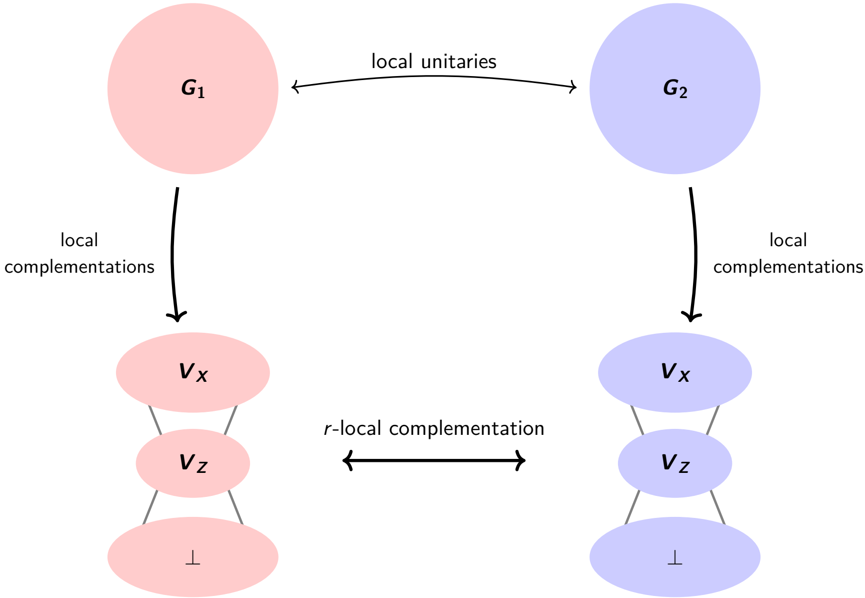

Local unitary equivalence of small graph states

Theorem (C, Perdrix, 2025)

Two graph states ***on at most 31 qubits*** *are local unitary equivalent if and only if the two corresponding graphs are related by a sequence of* ***2-local complementations****.*

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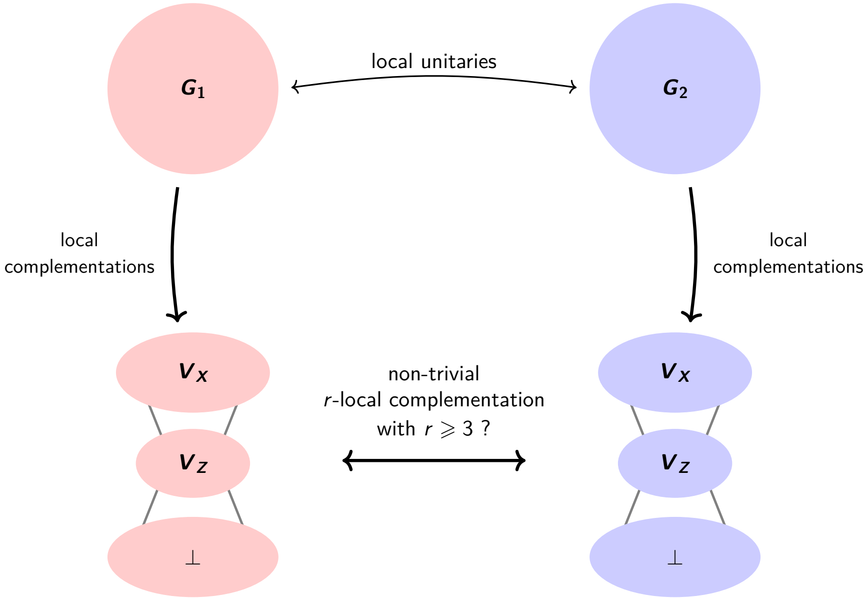

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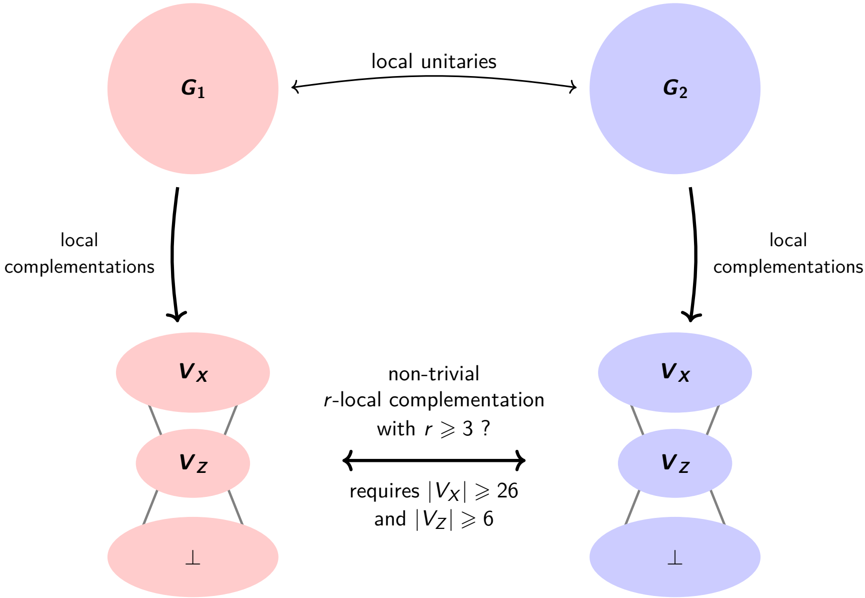

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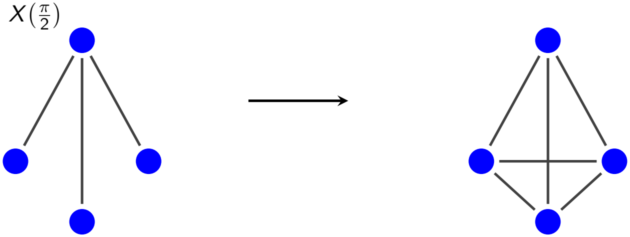

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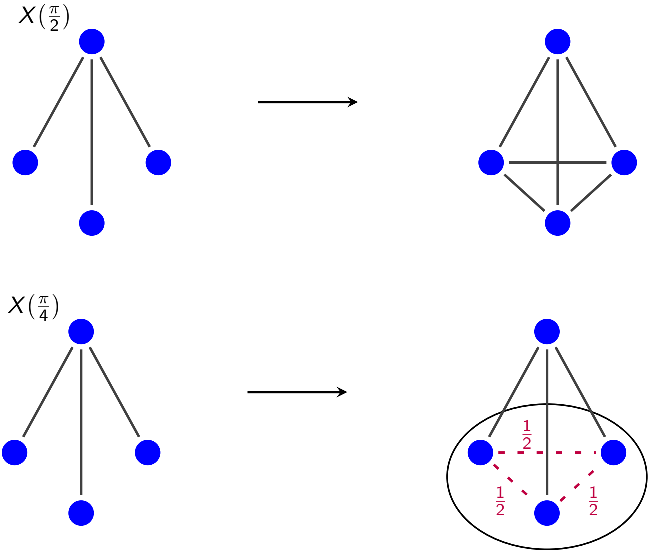

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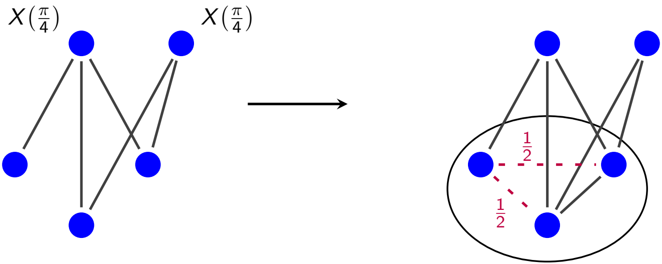

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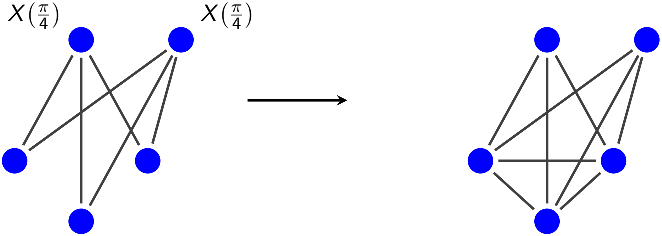
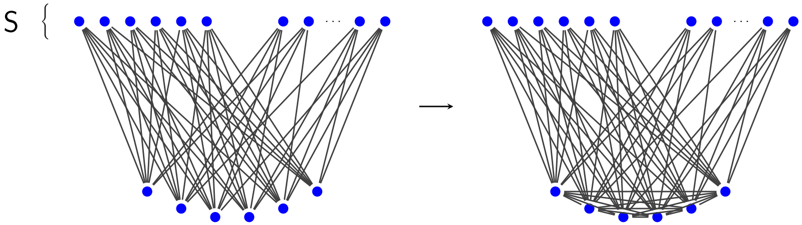


2-local complementation

2-local complementation is applied over a **set** S **of vertices**.



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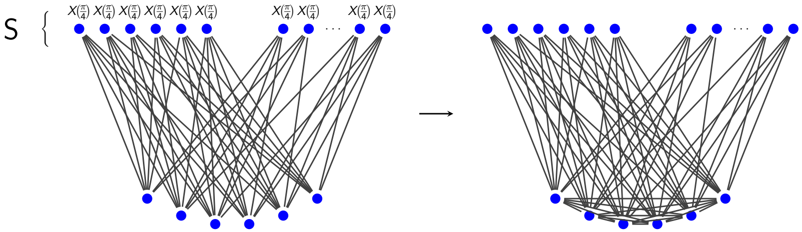

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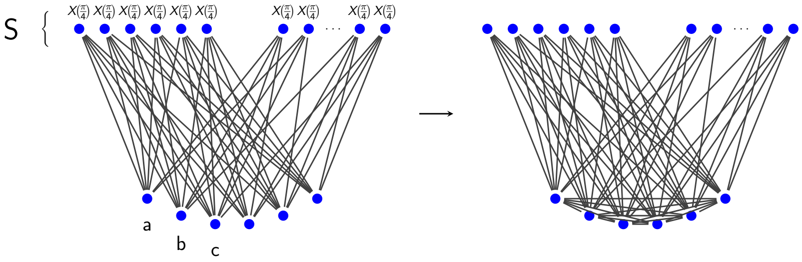

- $\forall a, b \notin S$, $|N(a) \cap N(b) \cap S| = 0 \mod 2$.
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$\rightarrow$ If every 2-local complementation on $\leqslant 26$ qubits can be implemented by local complementations, **the 27-qubit counter-example to the LU-LC conjecture is minimal**.

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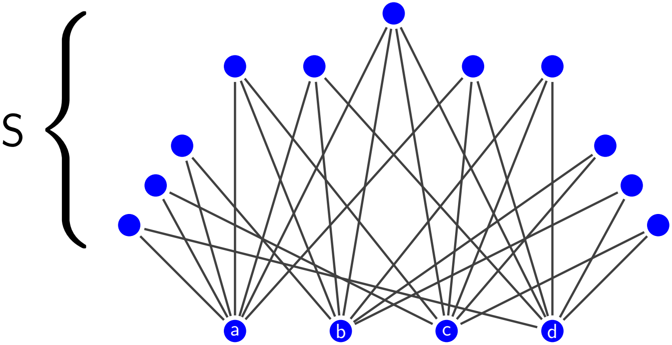

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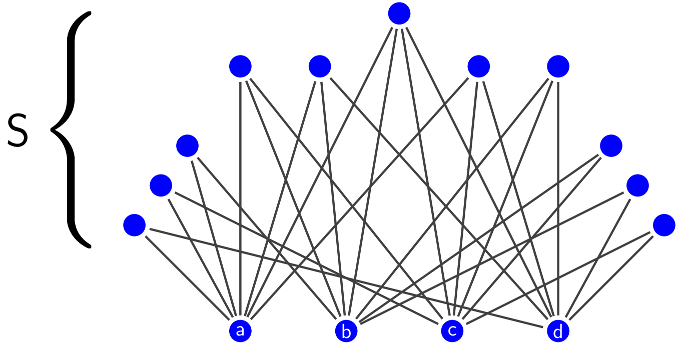

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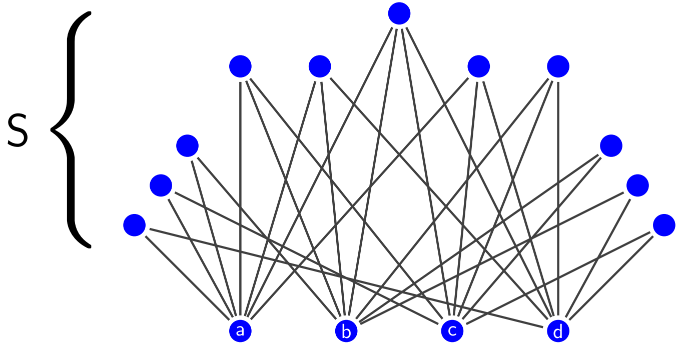

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Problem: still $> 10^{40}$ possible 2-local complementations on graph states with up to 26 qubits.

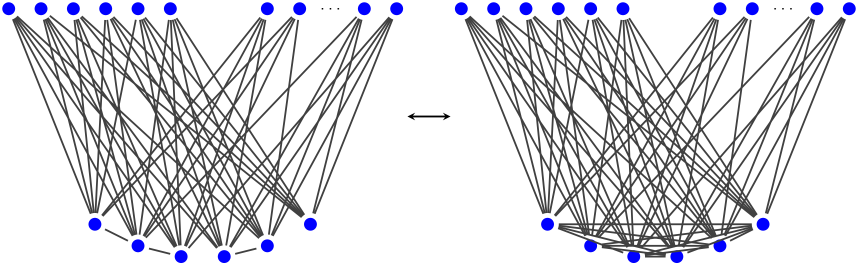
Reduction

Lemma (C., 2026)

*For a **minimal** counter-example to the LU-LC conjecture we can assume:*

- *Bipartite graph;*
- *No vertex of degree 1;*
- *No twins.*

Proof idea: bipartite graph



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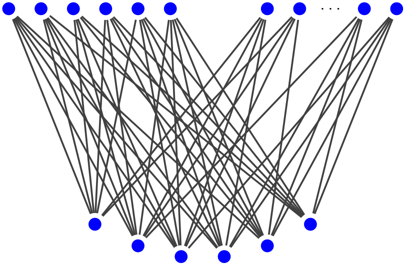
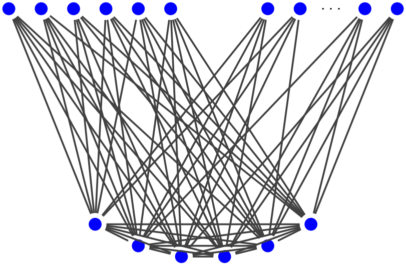
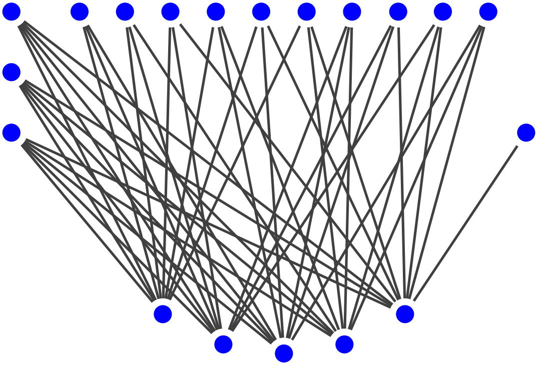


$\longleftrightarrow$



Proof idea: no vertex of degree 1



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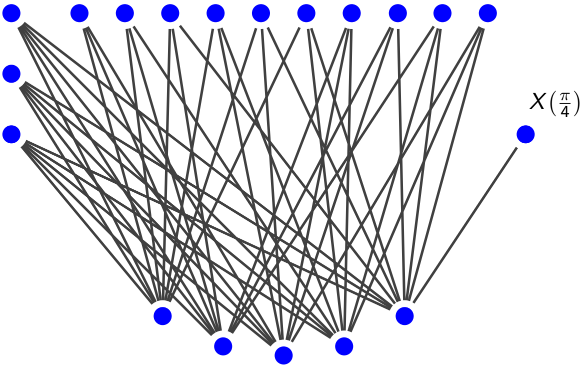

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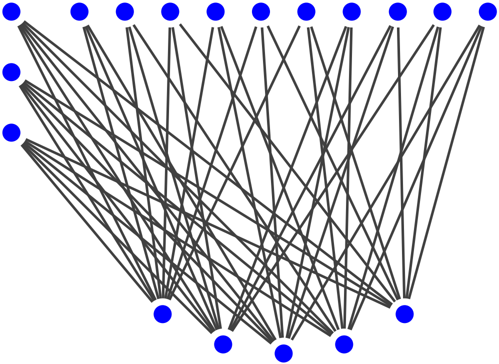

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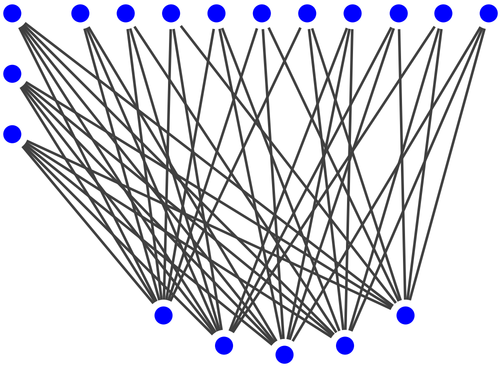

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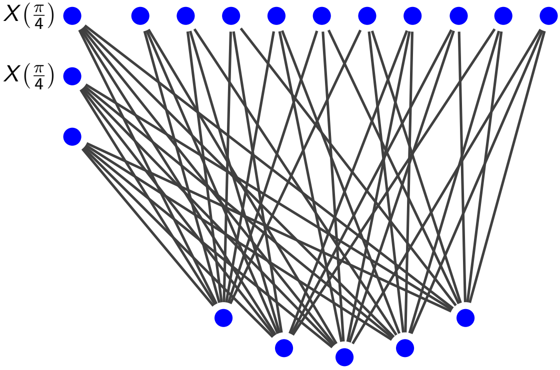

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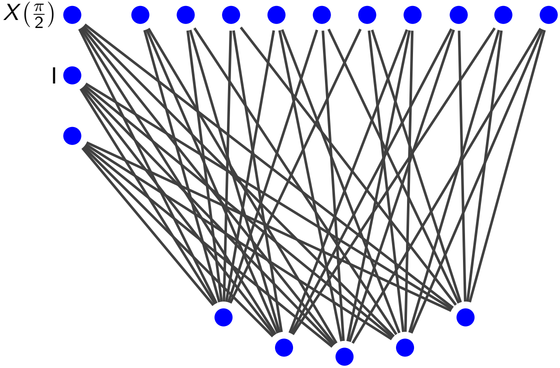

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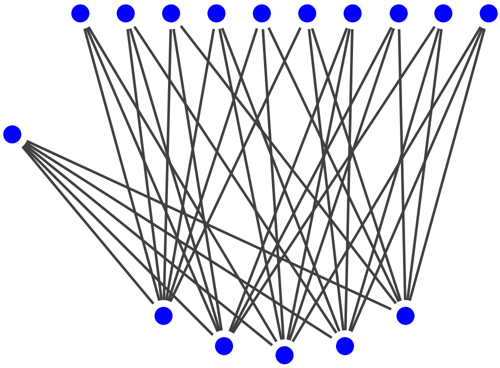

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Problem: still $> 10^{40}$ possible 2-local complementations on graph states with up to 26 qubits.

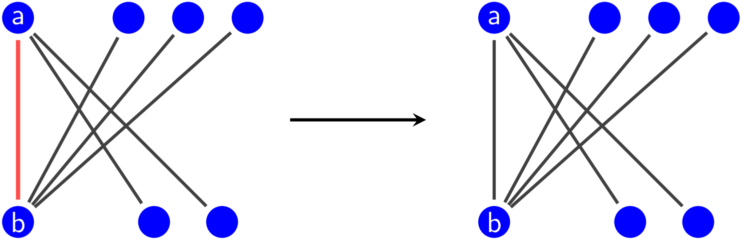
$\rightarrow$ one last ingredient.

Pivoting

Pivoting on edge *ab*

$=$

local complementations on $a \to$ on $b \to$ on a



Pivoting

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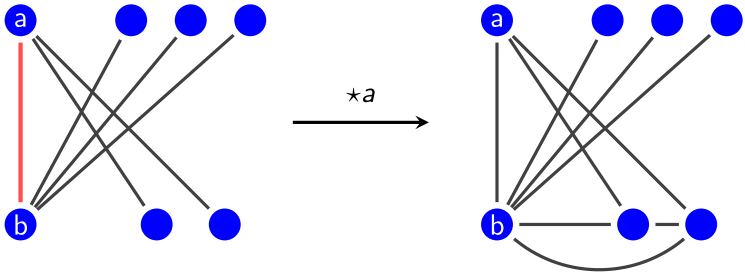

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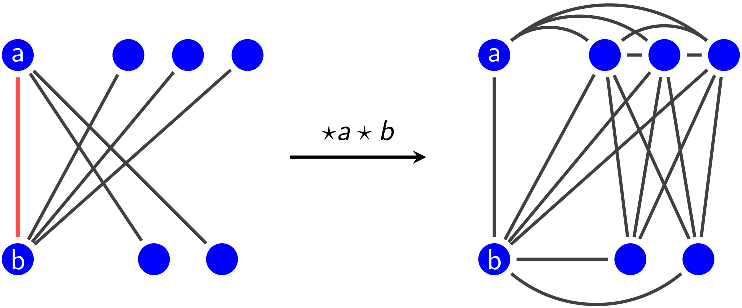

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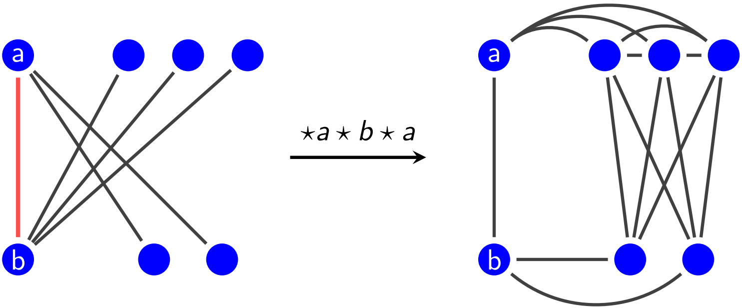

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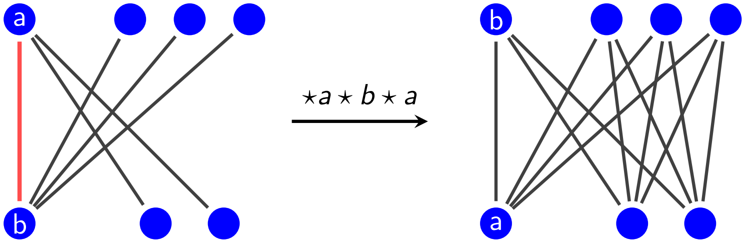
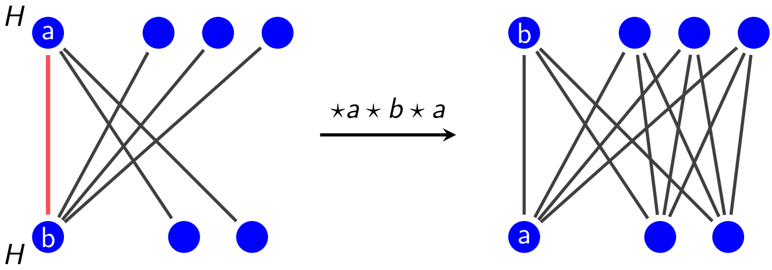


Pivoting

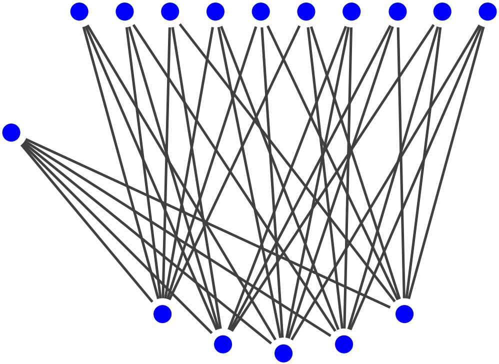
Pivoting on edge ab

$=$

local complementations on $a \rightarrow$ on $b \rightarrow$ on a



Pivoting preserves counter-examples



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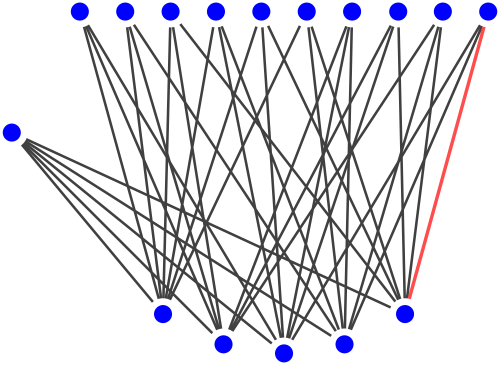

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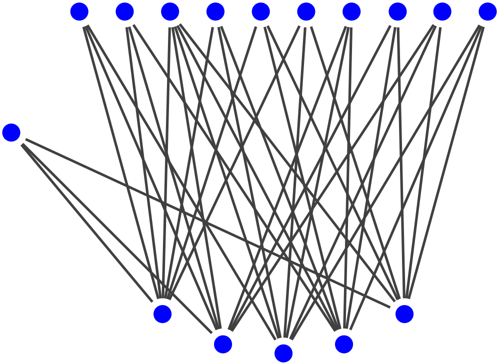

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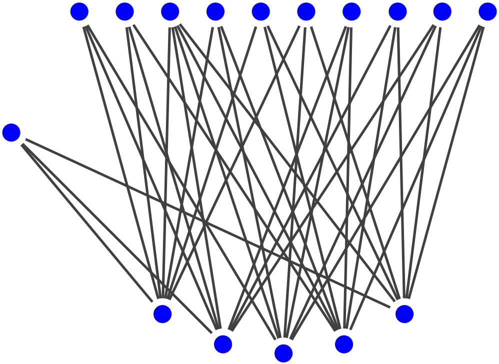
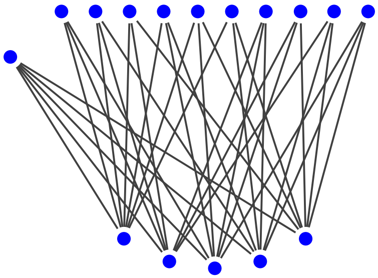


→ If a graph state is part of a counter-example to the LU-LC conjecture, every graph in the **pivoting orbit** is also part of a counter-example to the LU-LC conjecture.

Using pivotings

Proposition (informal)

For any class of counter-examples to the LU-LC conjecture, by means of pivotings, you can always find a representative ***with only vertices of odd degree****.*



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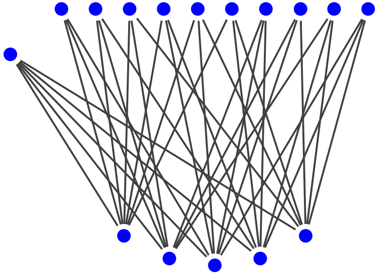
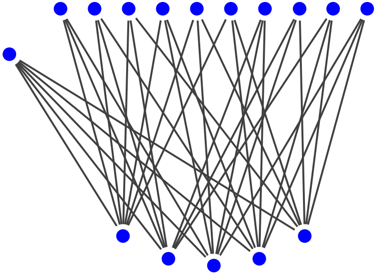


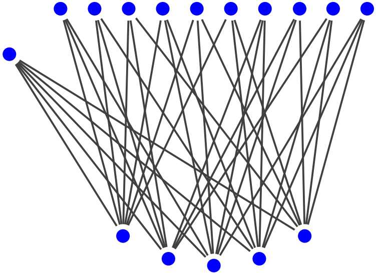
Very restrictive!

Connection with quantum error-correction



$$\longleftrightarrow \quad \begin{bmatrix} 1&1&1&1&1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0&1&1&1&0\\ 1&1&0&0&1&1&0&1&1&0&1\\ 1&0&1&0&1&0&1&1&0&1&1\\ 1&0&0&1&0&1&1&0&1&1&1 \end{bmatrix}$$

Connection with quantum error-correction



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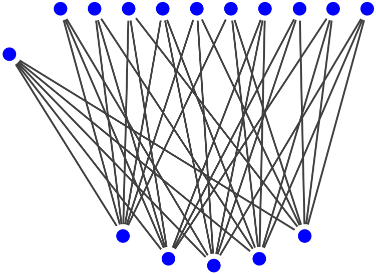
The matrix is **triorthogonal**: the product of every 2/3 rows has even Hamming weight.

Connection with quantum error-correction



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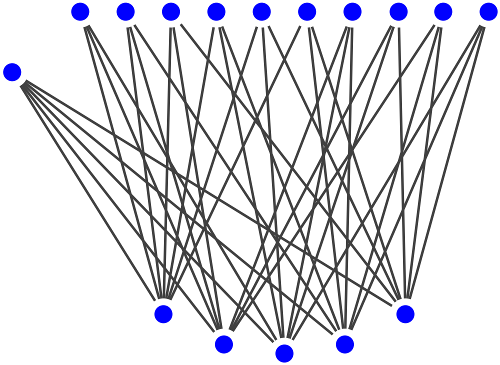
The matrix is **triorthogonal**: the product of every 2/3 rows has even Hamming weight.

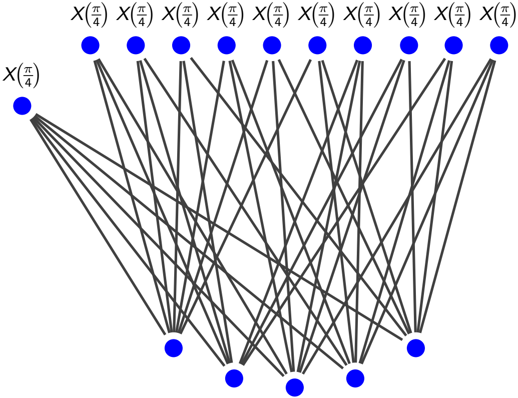
$\rightarrow$ Correspondence between 2-local complementations and triorthogonal codes, which have been classified[1].

[1] Nezami & Haah, Classification of small triorthogonal codes, PRA 2022

A 16-qubit candidate



A 16-qubit candidate



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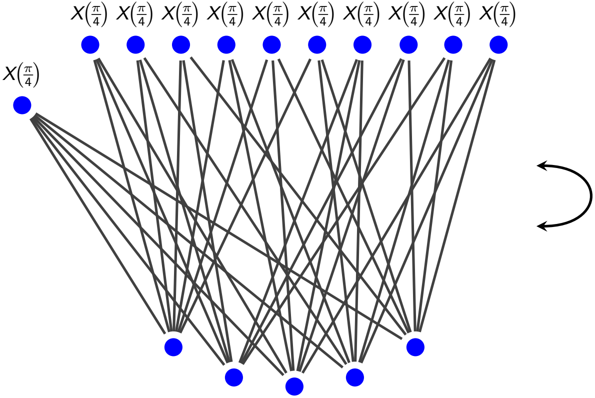

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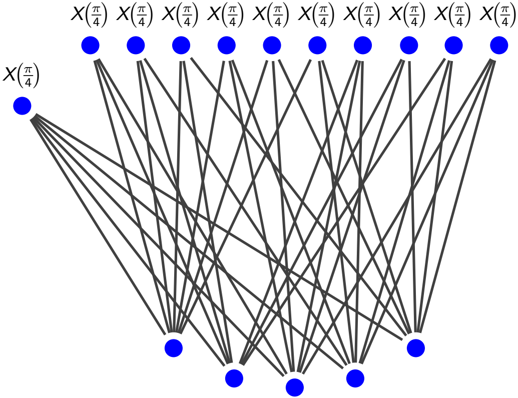
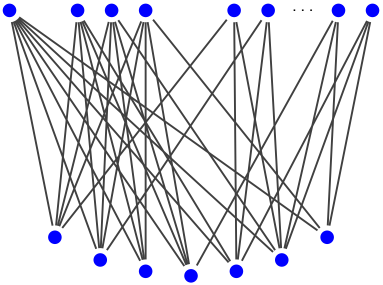
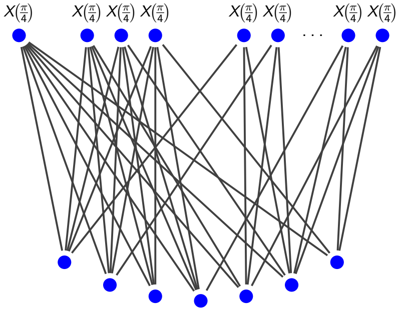


Not a counter-example to the LU-LC conjecture!

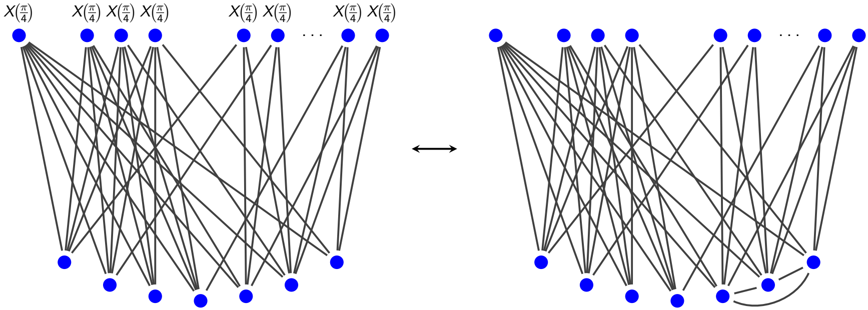
A 24-qubit candidate



A 24-qubit candidate



A 24-qubit candidate



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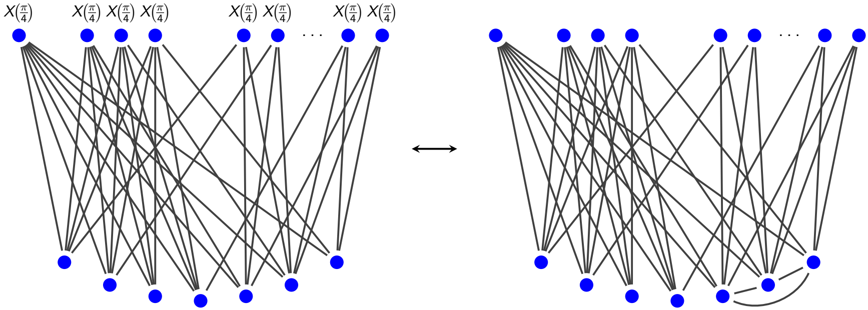
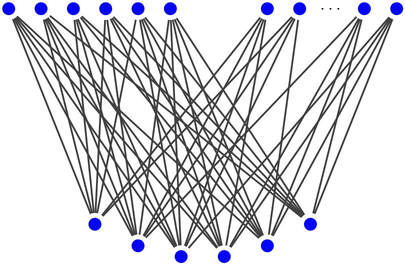


Not a counter-example to the LU-LC conjecture!

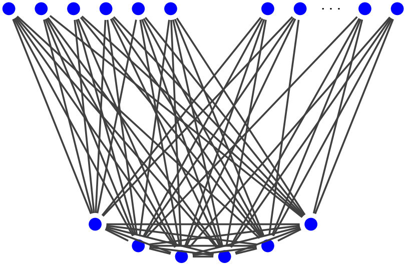
LU=LC up to 26 qubits

Theorem (C., 2026)

LU-equivalence and LC-equivalence coincide for graph states up to 26 qubits. ⇔ *The 27-vertex counterexample to the LU-LC conjecture is minimal.*



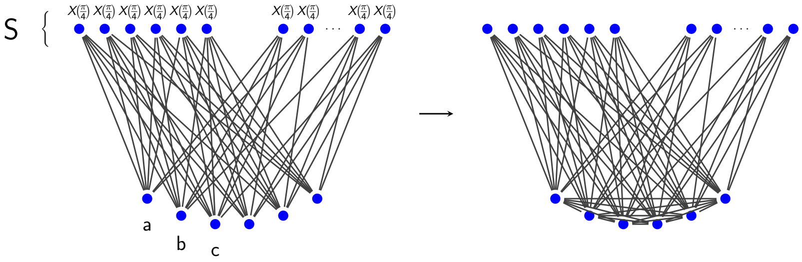
$\longleftrightarrow$



Beyond 2-local complementation

2-local complementation

2-local complementation is applied over a **set** S **of vertices**.



- $\forall a, b \notin S$, $|N(a) \cap N(b) \cap S| = 0 \mod 2$.
- $\forall a, b, c \notin S$, $|N(a) \cap N(b) \cap N(c) \cap S| = 0 \mod 2$.

r-local complementation

In general, r-local complementation is implemented with unitaries

$$X\left(\frac{\pi}{2^r}\right)$$

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Theorem (C, Perdrix, 2025)

*Two graph states are local unitary equivalent if and only if the two corresponding graphs are related by r-**local complementations** for some integer r.*

r-local complementation

In general, r-local complementation is implemented with unitaries

$$X\left(\frac{\pi}{2^r}\right)$$

Theorem (C, Perdrix, 2025)

*Two graph states are local unitary equivalent if and only if the two corresponding graphs are related by r-**local complementations** for some integer r.*

Theorem (C, Perdrix, 2025)

*Let r be a fixed integer. Two graph states are equivalent up to local unitaries **in the level** $r+1$ **of the Clifford hierarchy** if and only if the two corresponding graphs are related by r-local complementations.*

(Single-qubit) Clifford hierarchy

Level 1 : Paulis $\rightarrow$ X,Y,Z

(Single-qubit) Clifford hierarchy

$$\text{Level 1 : Paulis} \rightarrow \qquad \text{X,Y,Z}$$

$$\text{Level 2 : Clifford} \rightarrow \qquad C \text{ Pauli } C^{\dagger} = \text{Pauli}$$

(Single-qubit) Clifford hierarchy

Level 1 : Paulis $\rightarrow$ X,Y,Z

Level 2 : Clifford $\rightarrow$ C Pauli $C^\dagger$ = Pauli

Level 3 : U Pauli $U^\dagger$ = Clifford

(Single-qubit) Clifford hierarchy

$$\text{Level 1}: \text{Paulis} \rightarrow \qquad \text{X,Y,Z}$$

$$\text{Level 2}: \text{Clifford} \rightarrow \qquad C \text{ Pauli } C^{\dagger} = \text{Pauli}$$

$$\text{Level 3}: \qquad U \text{ Pauli } U^{\dagger} = \text{Clifford}$$

$$\text{Level 4}: \qquad U \text{ Pauli } U^{\dagger} = (\text{Level 3})$$

(Single-qubit) Clifford hierarchy

Level 1 : Paulis $\rightarrow$ X,Y,Z

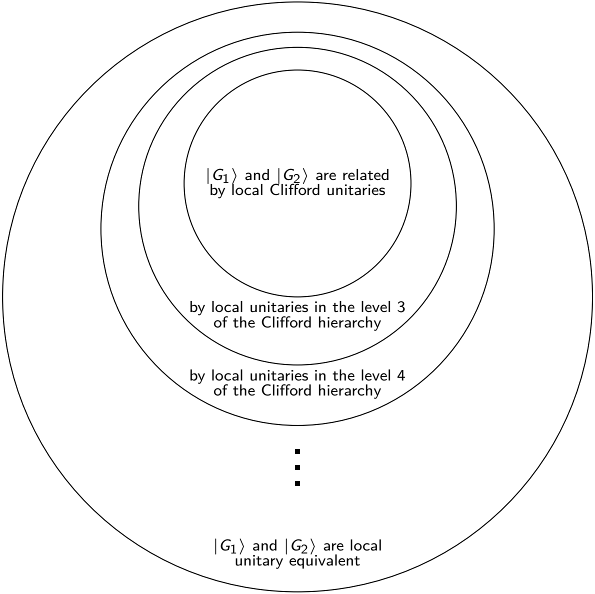
Level 2 : Clifford $\rightarrow$ C Pauli $C^\dagger$ = Pauli

Level 3 : U Pauli $U^\dagger$ = Clifford

Level 4 : U Pauli $U^\dagger$ = (Level 3)

an so on...

An infinite hierarchy of local equivalences



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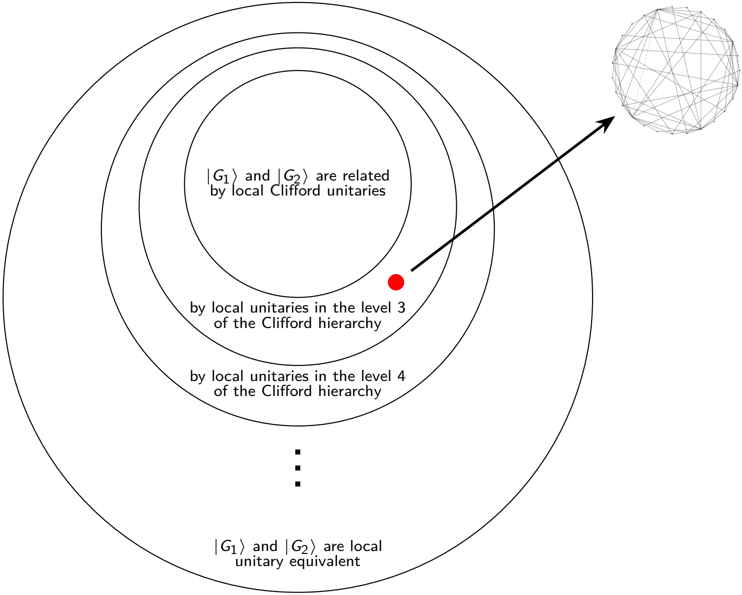

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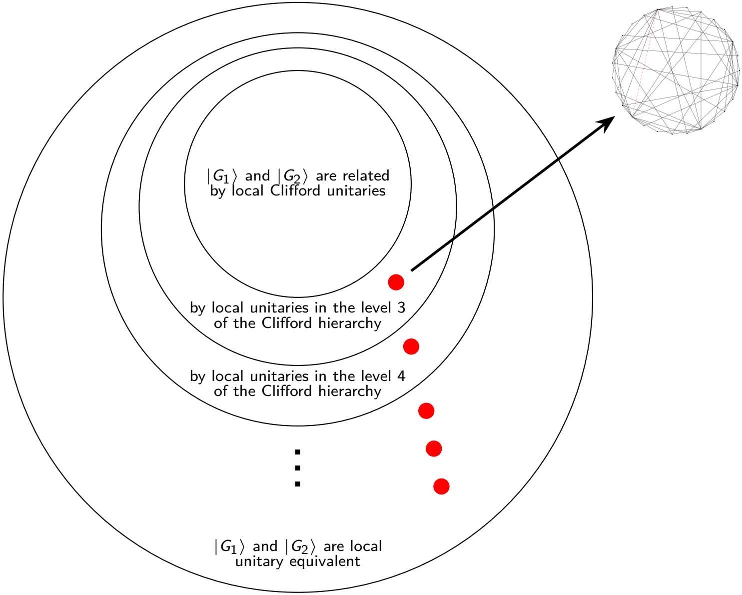

Minimal "generalized" counter-examples

Definition

Given an integer r, the number $c(r)$ is the minimal number of qubits of a pair of graph states related by local unitaries in the level $r+1$ of the Clifford hierarchy but not by local unitaries in the level r.

Minimal "generalized" counter-examples

Definition

Given an integer r, the number $c(r)$ is the minimal number of qubits of a pair of graph states related by local unitaries in the level $r+1$ of the Clifford hierarchy but not by local unitaries in the level r.

Proposition

For every integer r, $c(r)$ is finite.

c(1)=?

$c(1)=?$

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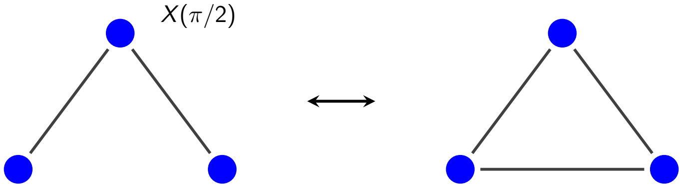
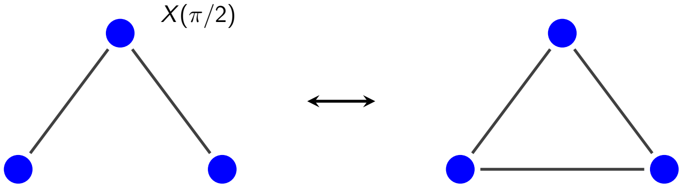
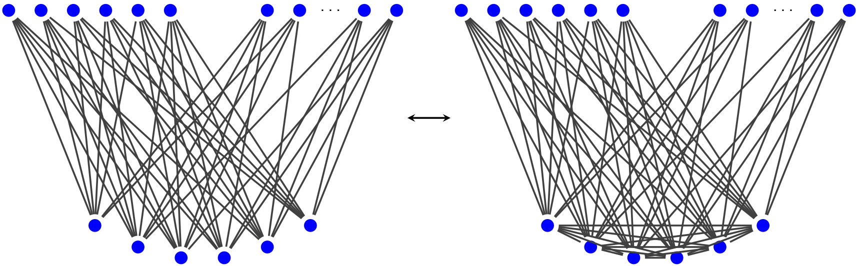


c(1)=?



$$c(1) = 3$$

c(2)=?



$$c(2) = 27$$

c(3) and beyond?

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$$32 \leqslant c(3) \leqslant 5020$$

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In general:

$$2^{r+3} \leqslant c(r) \leqslant$$
$$2^r + 2^{\lceil \log_2(r+2) \rceil} - 1 + \binom{2^r + 2^{\lceil \log_2(r+2) \rceil} - 1}{2^{\lceil \log_2(r+2) \rceil} - 2}$$

c(3) and beyond?

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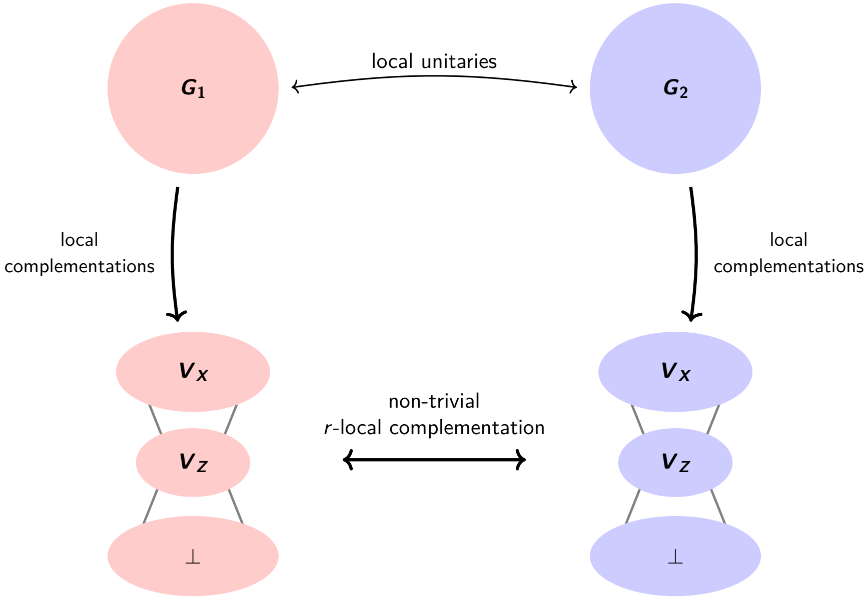
In general:

$$2^{r+3} \leqslant c(r) \leqslant 2^r + 2^{\lceil \log_2(r+2) \rceil} - 1 + \binom{2^r + 2^{\lceil \log_2(r+2) \rceil} - 1}{2^{\lceil \log_2(r+2) \rceil} - 2} \sim 2^{r^2}$$

Proof sketch: lower bound



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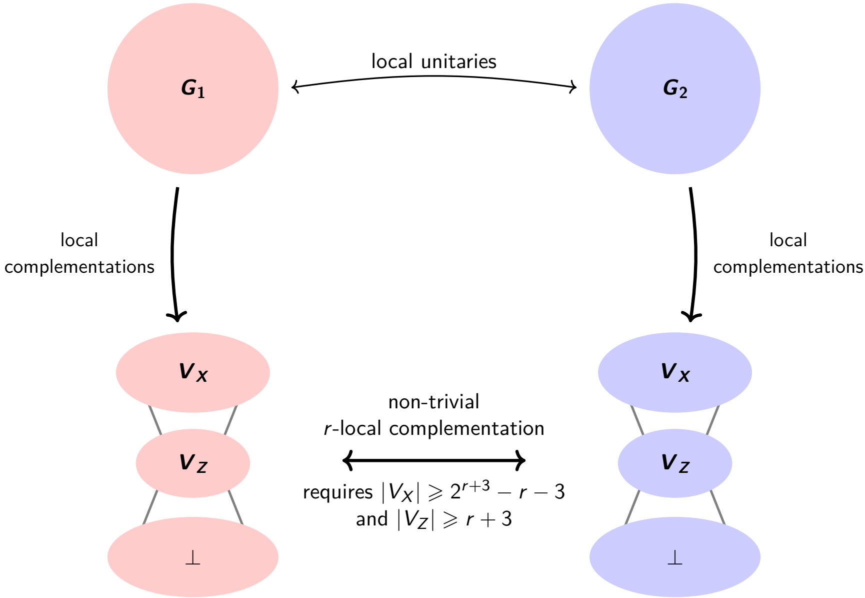
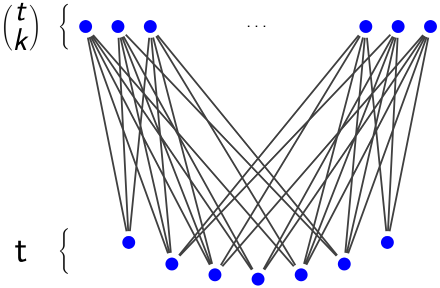


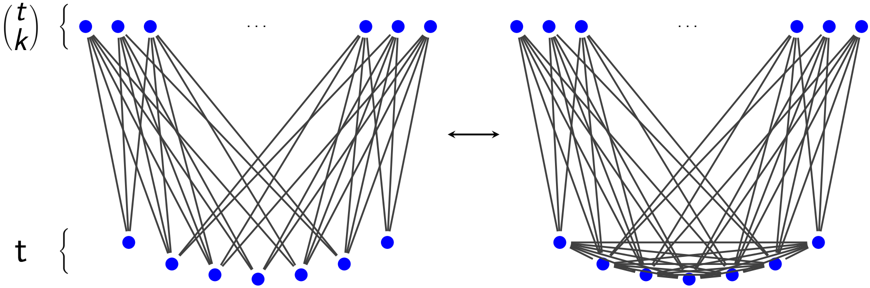
Proof sketch: upper bound

Graph defined by an integer t and an integer k.



Proof sketch: upper bound

Graph defined by an integer t and an integer k.



We set $t = 2^r + 2^{\lceil \log_2(r+2) \rceil} - 1$ and $k = 2^{\lceil \log_2(r+2) \rceil} - 2$.

Bounds

$$2^{r+3} \leqslant c(r) \leqslant$$
$$2^r + 2^{\lceil \log_2(r+2) \rceil} - 1 + \binom{2^r + 2^{\lceil \log_2(r+2) \rceil} - 1}{2^{\lceil \log_2(r+2) \rceil} - 2}$$
$$\sim 2^{r^2}$$

Summary

LU=LC up to 26 qubits

$\Leftrightarrow$

The 27-qubit counter-example to the LU-LC conjecture is minimal

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Future work:

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LU=LC up to 26 qubits

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- why 27?

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LU=LC up to 26 qubits

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Future work:

- why 27?
- Connection between r-local complementation and quantum error-correction

Summary

LU=LC up to 26 qubits

$\Leftrightarrow$

The 27-qubit counter-example to the LU-LC conjecture is minimal

Future work:

- why 27?
- Connection between r-local complementation and quantum error-correction
- Better bounds for $c(r)$ (currently $2^{r+3} \leqslant c(r) \leqslant \sim 2^{r^2}$)

Thanks

arXiv:2603.25219