Local equivalences of graph states

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Maria Waldrast
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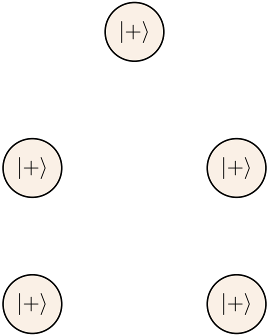
$\rightarrow$

Graph states, local unitary equivalence, local Clifford equivalence & local complementation

Graph states

A graph state is a quantum state represented by an undirected[1] and simple[2] graph. The vertices represent the qubits and the edges represent entanglement.



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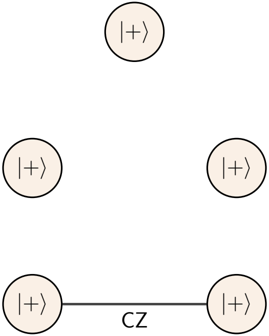

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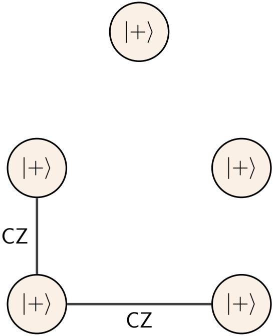

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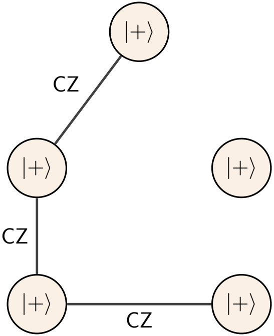

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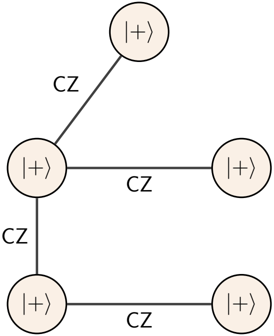

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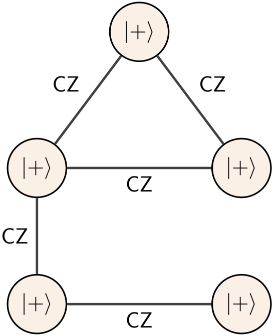

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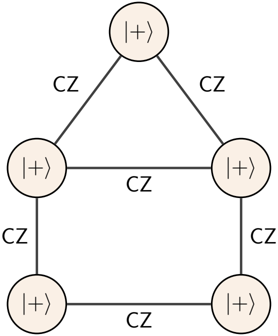

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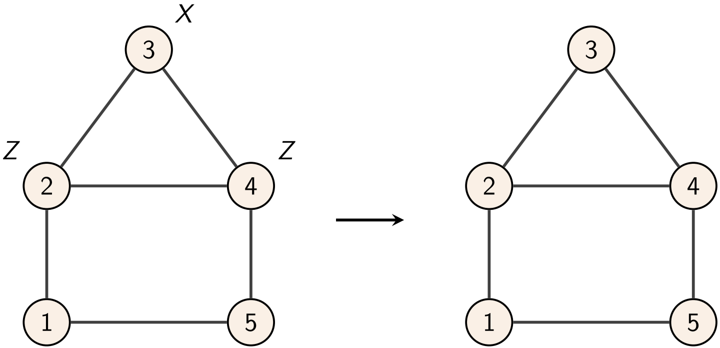
Stabilizer states

Graph states are a subfamilly of stabilizer states because for each vertex u, applying X on u and applying Z on the neighbours of u leaves the graph state invariant.



Entanglement of graph states

Graph states are useful entangled resources (MBQC, quantum networks...).
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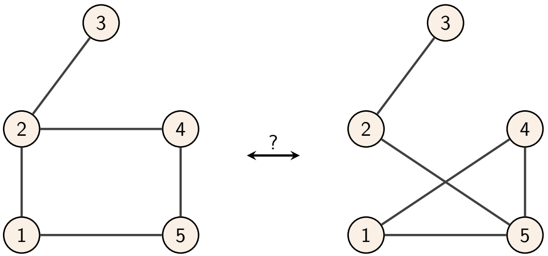

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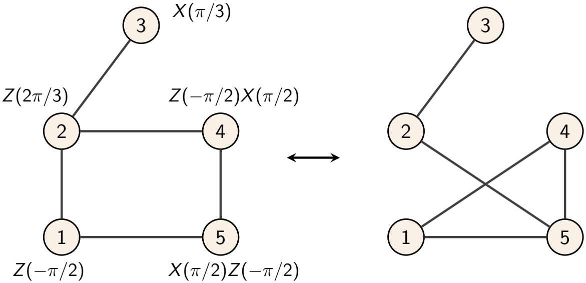
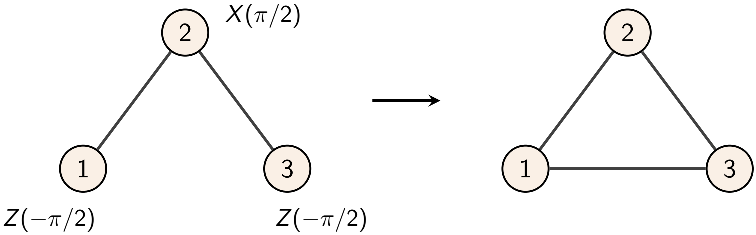


An easier subproblem: local Clifford equivalence

Two graph states are said **local Clifford equivalent** (or LC-equivalent) if they are related by unitaries in the local Clifford group.
Single-qubit Clifford group $= < H, Z(\pi/2) >$.



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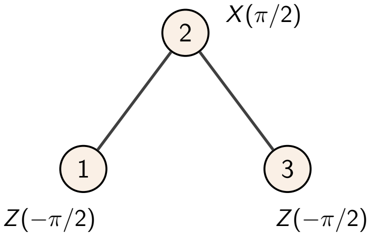


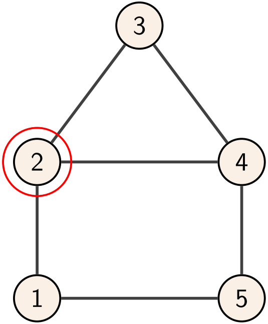
Theorem (Van den Nest, Dehaene, De Moor, 2004)

Two graph states are LC-equivalent iff the two corresponding graphs are related by **local complementations**.

Local complementation

Definition (Kotzig, 1966)

A local complementation on a vertex u consists in complementing the (open) neighbourhood of u.



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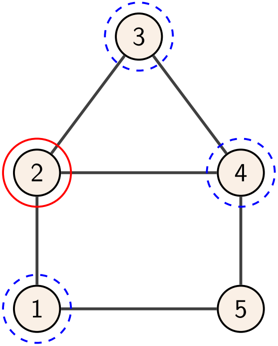

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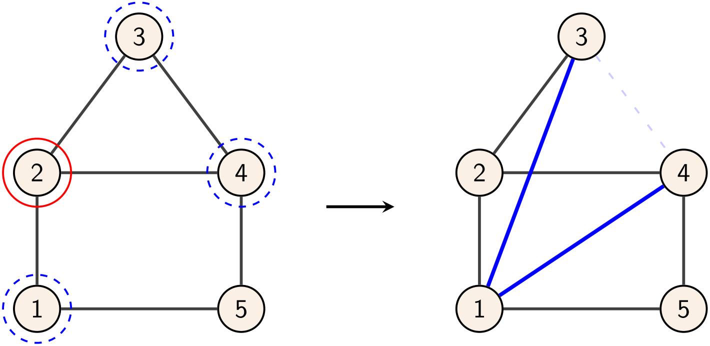

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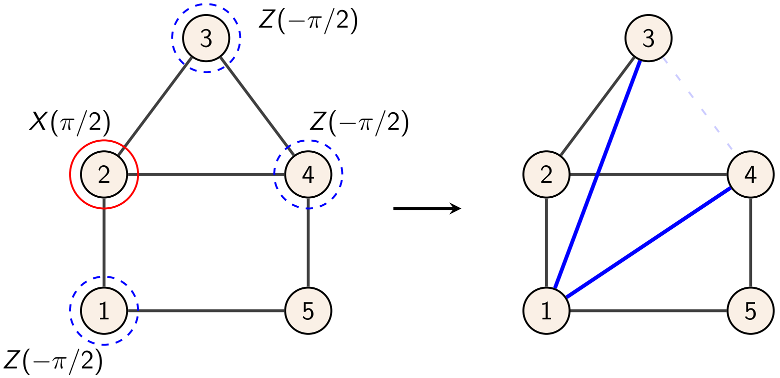

Algorithmic aspect of local Clifford equivalence

Proposition (Bouchet, 1991)

There exists an efficient algorithm to decide if two graphs are related by local complementations.

$\rightarrow$ algorithm to decide if two graph states are LC-equivalent.

Quick history of the LU=LC conjecture

The LU=LC conjecture

Formulated in the early 2000's.

Conjecture

LU=LC i.e. if two graph states are LU-equivalent (local unitaries) then they are LC-equivalent (local Clifford).

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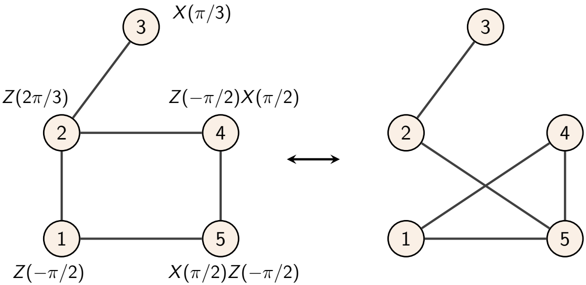

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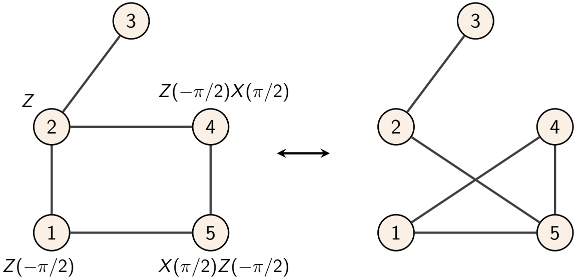

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LU $\neq$ LC, i.e. local unitary equivalence and local Clifford equivalence do **not** coincide.

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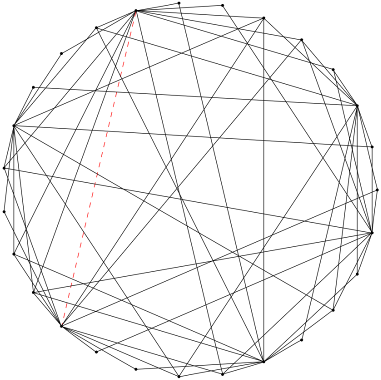
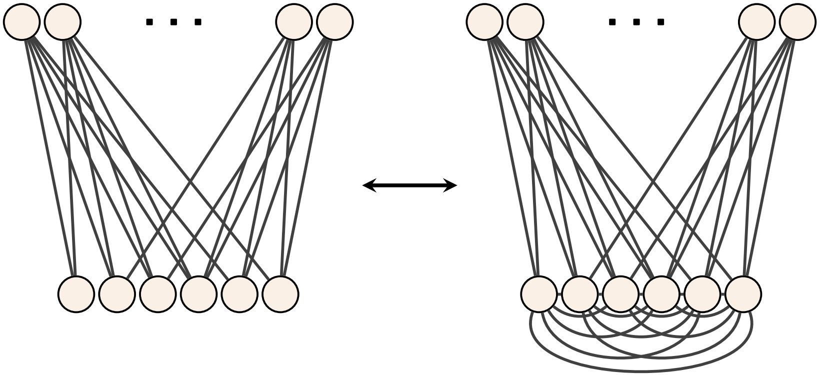
LU $\neq$ LC, i.e. local unitary equivalence and local Clifford equivalence do **not** coincide.$\rightarrow$ 27-qubit pair of graph states that are LU-equivalent but not LC-equivalent (Ji et al. 2008).



Another look at the 27-vertex counterexample

The 27-vertex counterexample is LC-equivalent to a prettier pair of graphs (Tsimakuridze, Gühne, 2017).



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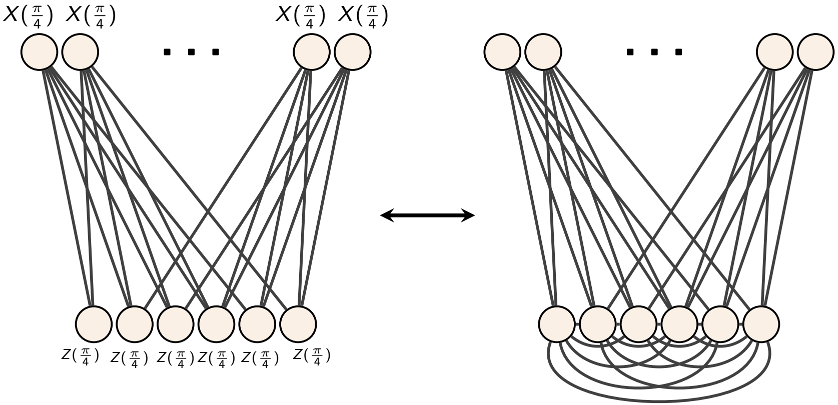

Other results after LU $\neq$ LC

Proposition (Sarvepalli, Raussendorf, 2010)

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- Graph states over at most 8 qubits (Cabello et al. 2009)

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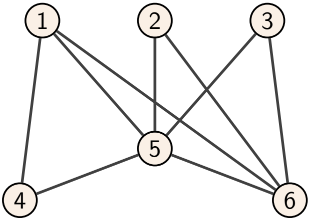
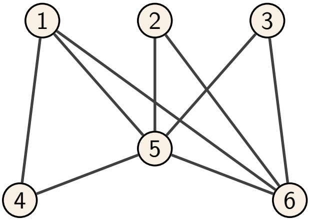
But what about LU-equivalence for **any** graph? Can we construct a graphical characterisation?

Generalizing local complementation to capture local unitary equivalence

A refinement of idempotent local complementations

A sequence of local complementations may leave the graph invariant.





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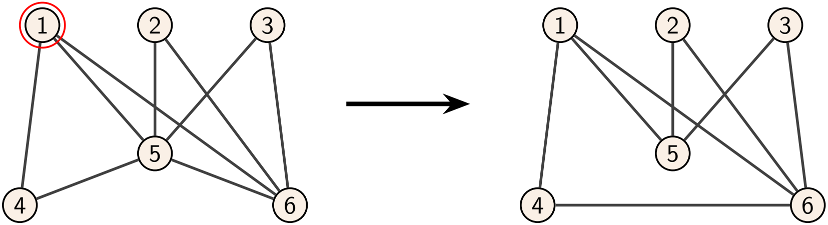

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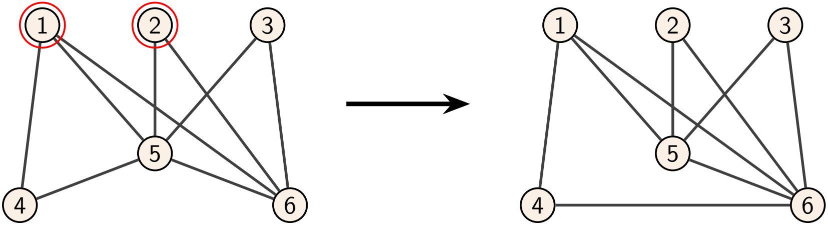

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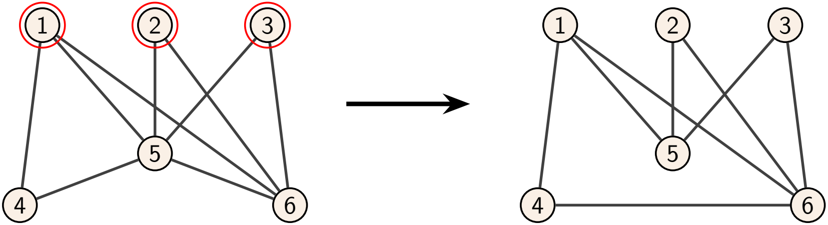

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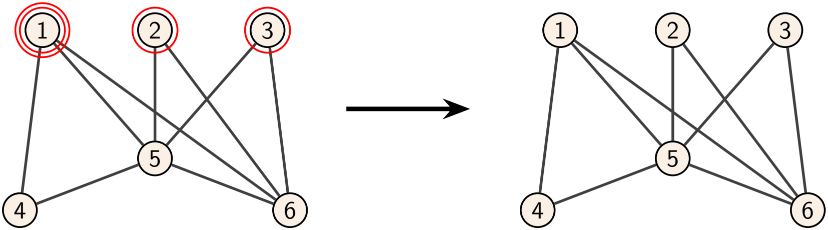

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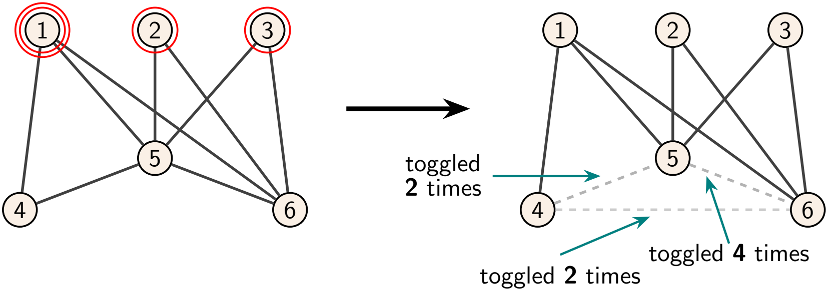

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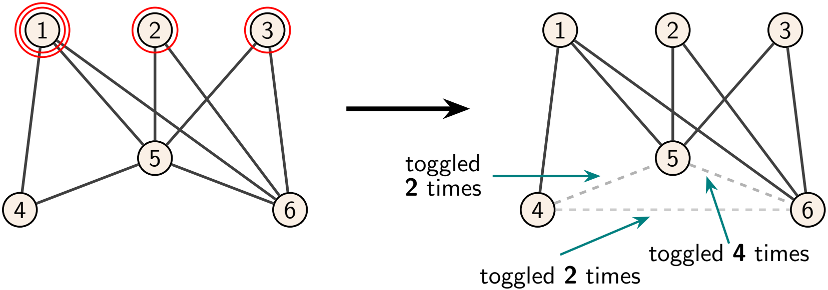

A **2-local complementation** consists in toggling every edge that was toggled 2 mod 4 times by the idempotent local complementations.
(There are also some additional conditions on the edges for the 2-local complementation the be valid.)

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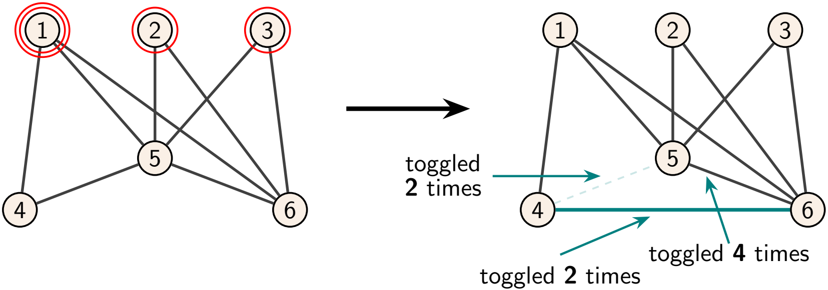

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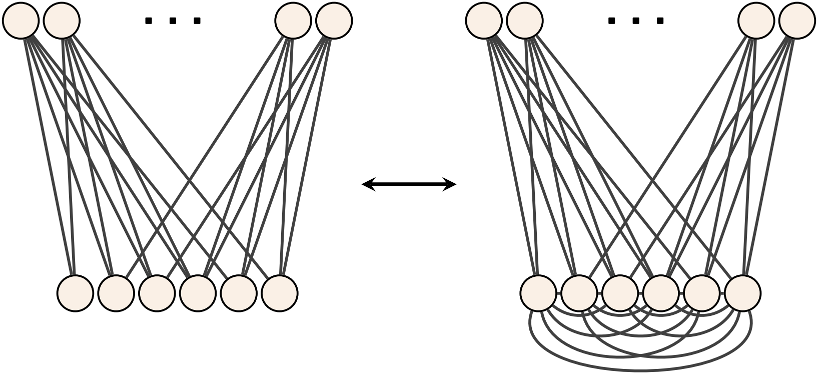

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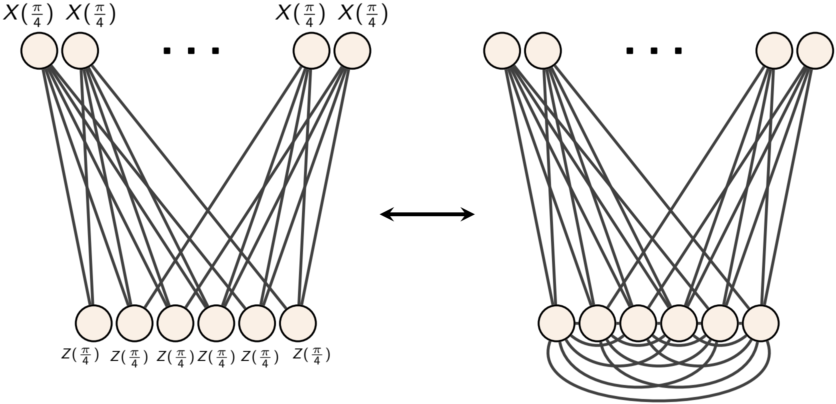

r-local complementation

3-local complementation is a refinement of idempotent 2-local complementation, and so on...
$\rightarrow$ Infinite family of graphical operations parametrised by an integer r:

r-local complementations

1-local complementation = local complementation.

Graphical characterization of entanglement

Recall: LC-equivalent $\Leftrightarrow$ related by local unitaries generated by finitely many H and $Z(\pi/2)$.
Define: **LC$_r$-equivalent** $\Leftrightarrow$ related by local unitaries generated by finitely many H and $Z(\pi/2^r)$.

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Theorem (C, Perdrix, 2025)

Two graph states are LC$_r$-equivalent iff the two corresponding graphs are related by r-local complementations.

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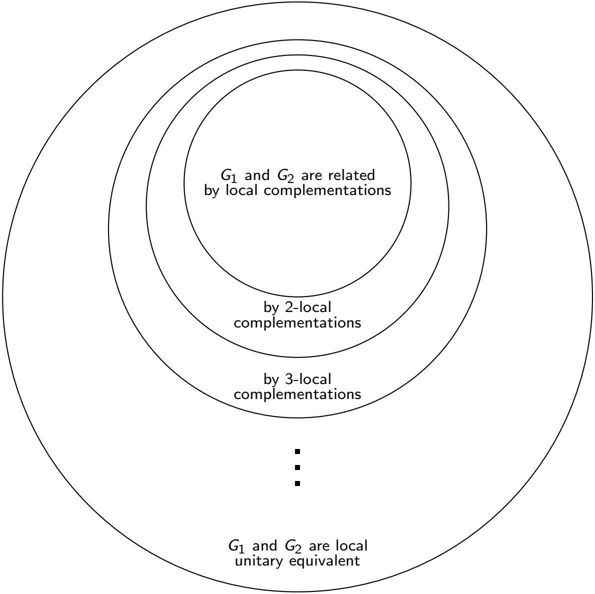
An infinite hierarchy of local equivalences



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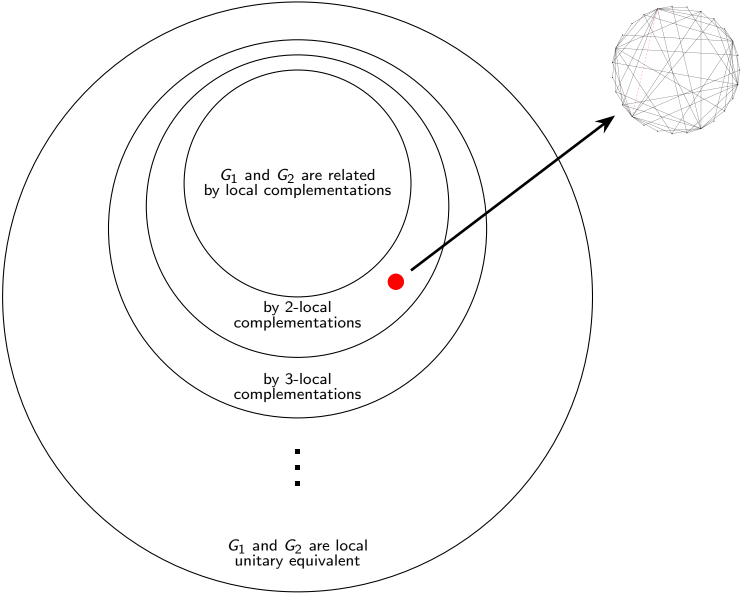

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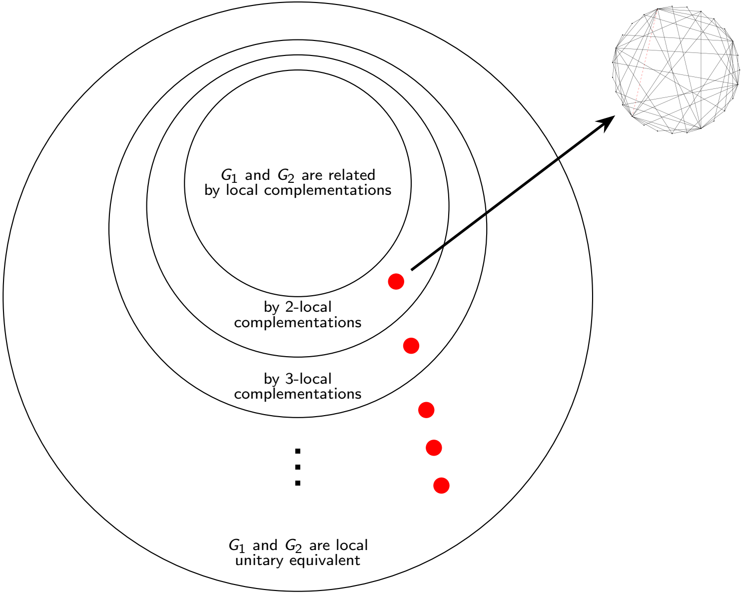
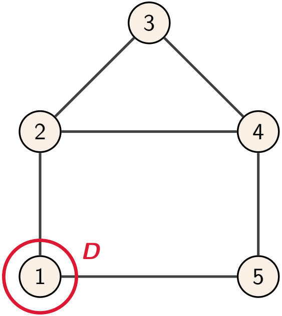


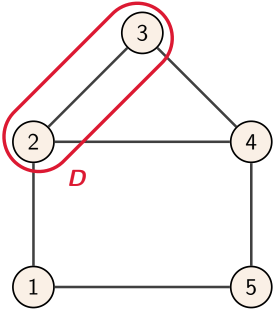
Proof that r-local complementation captures LU-equivalence

Minimal local sets

Definition (Odd neighbourhood)

Given a set of vertices D, the **odd neighbourhood** $Odd_G(D)$ of D is the set of vertices that are neighbours of an odd number of vertices in D.





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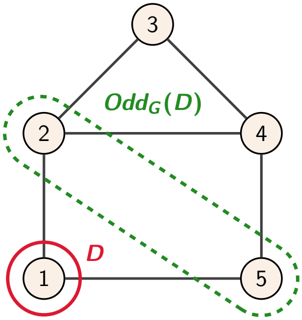

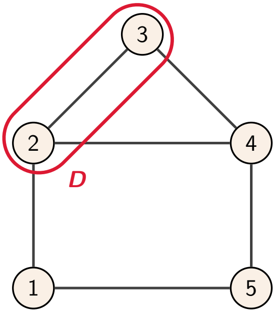

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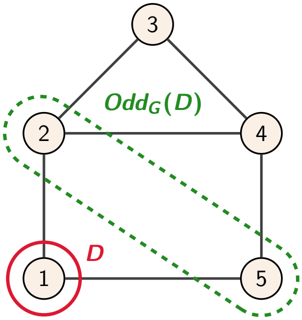
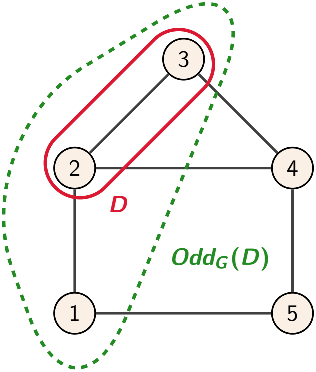
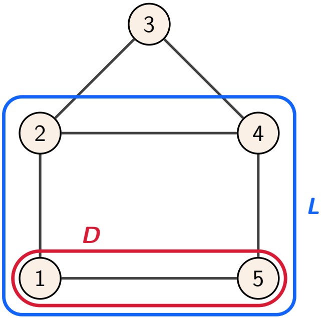




Minimal local sets

Definition

A **local set** is a non-empty vertex set of the form $L = D \cup Odd_G(D)$.
A **minimal local set** is a local set that is minimal by inclusion (i.e it doesn't strictly contain another local set).



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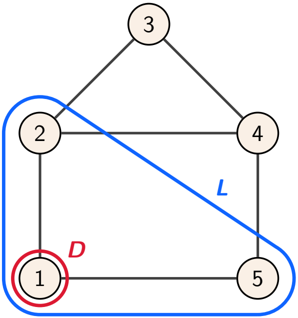

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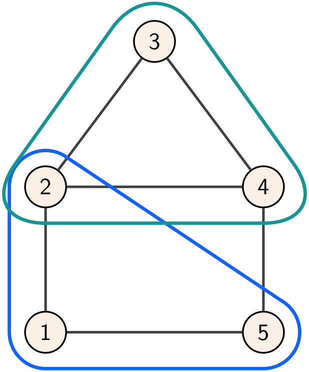
Minimal local sets $\Leftrightarrow$ stabilizers of minimal support.

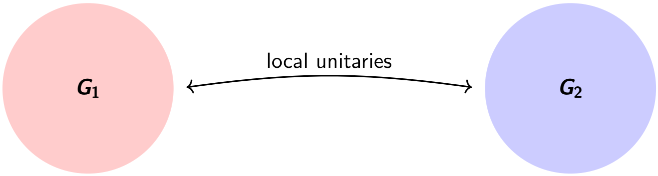
Minimal local sets carry information on the possible local unitaries that maps graph states to other graph states.

Minimal local sets cover any graph

Theorem (C, Perdrix, 2024)
Each vertex of a graph is covered by at least one minimal local set.



Proof sketch: Standard form for graph states



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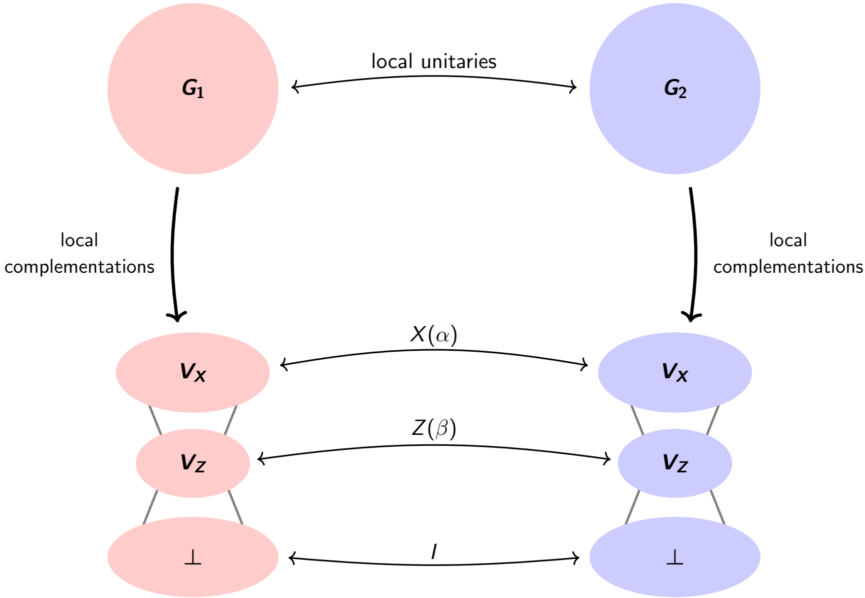

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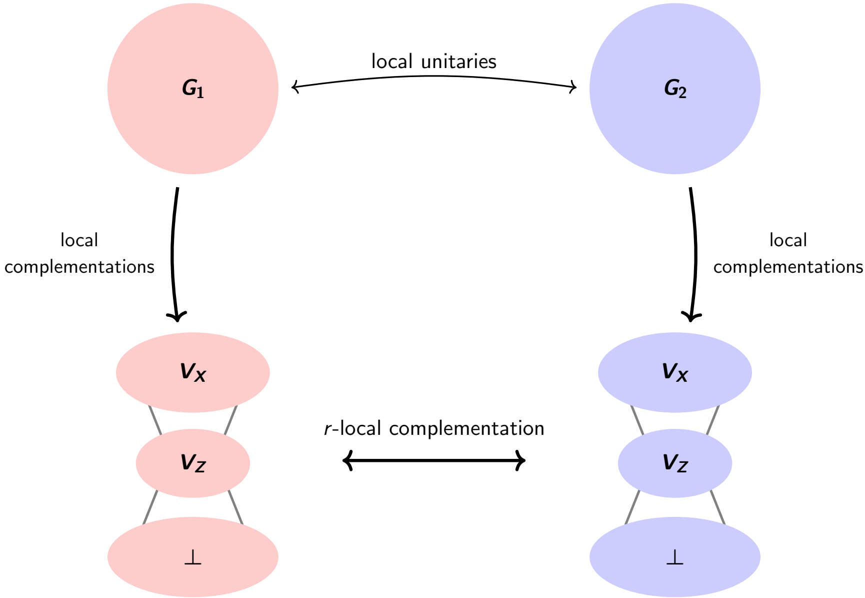

Some areas of research on graph states

Algorithms for LU-equivalence

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Question

Does there exist an efficient algorithm that decides if two graph states are LU-equivalent?

Algorithms for LU-equivalence

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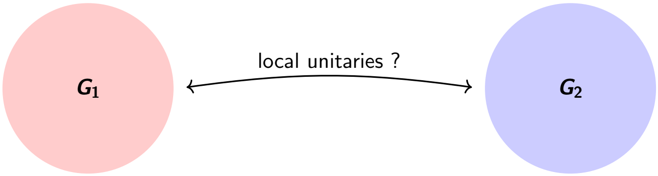
Does there exist an efficient algorithm that decides if two graph states are LU-equivalent?

Some progress: a quasi-polynomial algorithm.

Theorem (C, Perdrix, 2025)

There exists an algorithm that decides if two graph states are LU-equivalent with runtime $n^{log_2(n)+O(1)}$.

The algorithm



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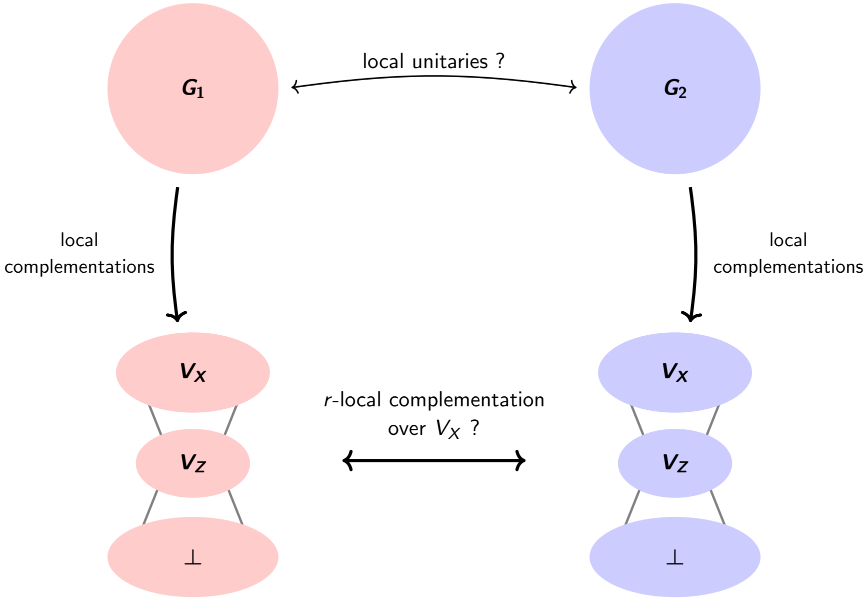

LU=LC up to 26 qubits?

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Conjecture

LU=LC for graph states up to 26 qubits.

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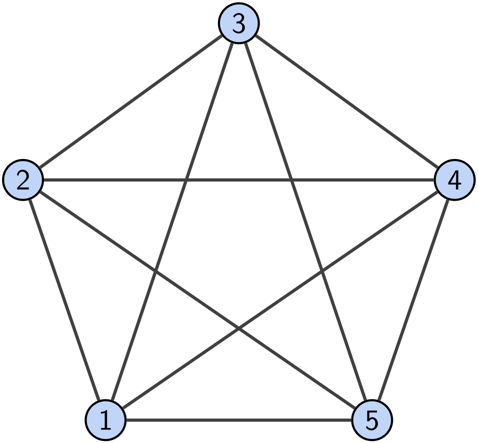
Some progress:

Proposition

LU=LC for graph states up to 19 qubits.

Optimisation of graph state preparation

Why minimizing edge-count matters



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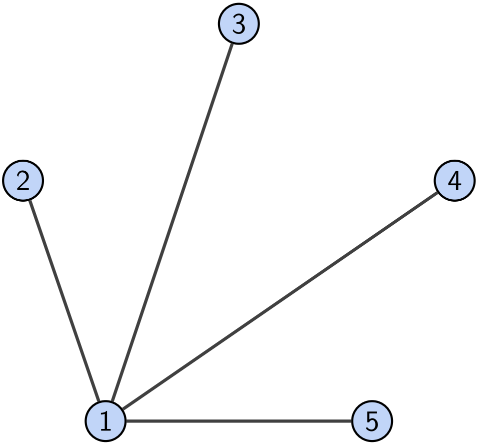

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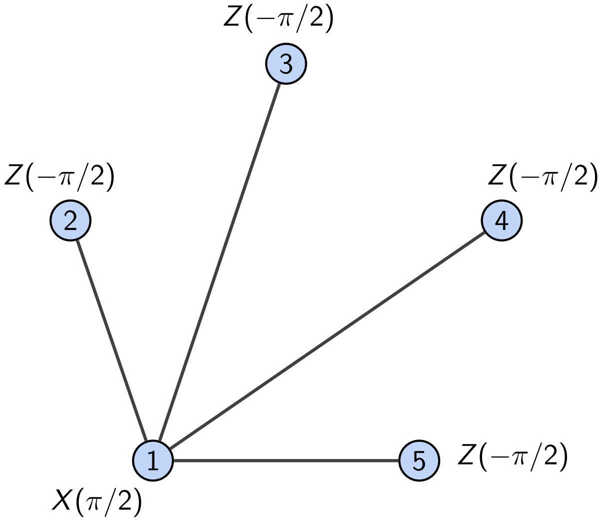

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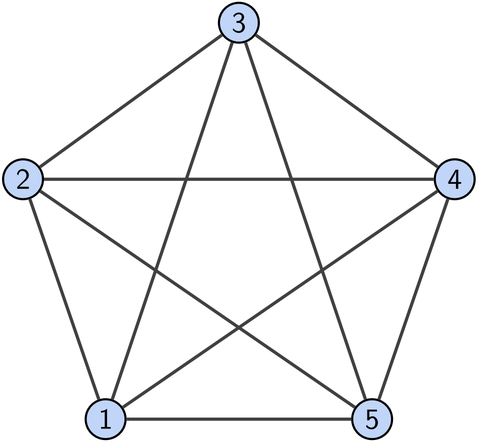

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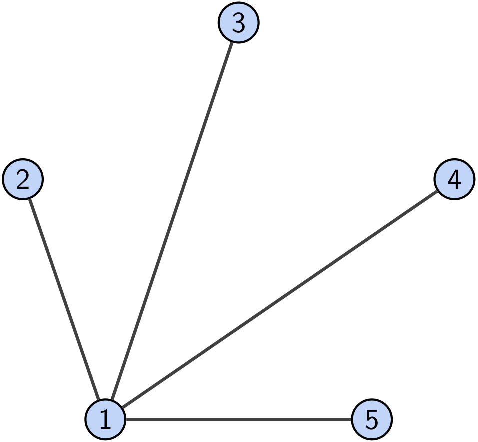

Optimisation of graph state preparation

There are many ways of formalizing the problem of preparing a graph state optimally:

[1] Kumabe, Mori, Yoshimura, Complexity of graph-state preparation by Clifford circuits, 2025

[2] Davies, Jena, Preparing graph states forbidding a vertex-minor, 2025

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- Other implementation-specific operations are allowed, like fusion for photonic graph states[4];
- Local unitaries (not just local Clifford) i.e. r-local complementations are allowed.

---

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Can r-local-complementation help?

Question

Can local unitary gates beyond local Clifford gate (i.e. r-local complementations) help reduce the edge-count?

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Can local unitary gates beyond local Clifford gate (i.e. r-local complementations) help reduce the edge-count?

More generally:

Question

How much richer is the orbit by r-local complementation compared to the orbit by local complementation.

Universality and classical simulation

Question

Which classes of graph states are universal for MBQC? For which classes of graph states is MBQC classically simulable?

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MBQC on graph states with bounded **rank-width** *is classically simulable.*

Universality and classical simulation

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Which classes of graph states are universal for MBQC? For which classes of graph states is MBQC classically simulable?

Proposition (Van den Nest, Dür, Vidal, Briegel 2007)

MBQC on graph states with bounded **rank-width** *is classically simulable.*

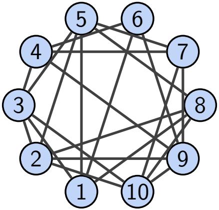
Conjecture (Geelen)

MBQC can be simulated classically for (non-trival) classes of graphs that are closed under local complementation and vertex deletions.

Efficient universality of random graph states

Question

Are random graph states efficiently universal?



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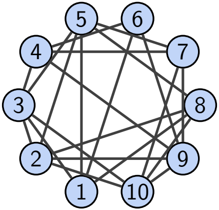

- Random **quantum states** are not efficiently universal because their **geometric measure of entanglement** is too high.[1]

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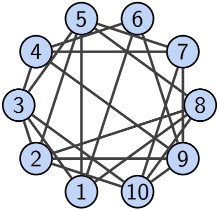

- Random **quantum states** are not efficiently universal because their **geometric measure of entanglement** is too high.[1]
- The geometric measure of entanglement of random **graph states** is too low to use the same argument.[2]

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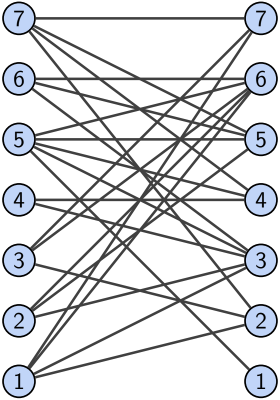
Efficient universality of random bipartite graph states



With high probability, a cluster state with $\sqrt{n}$ qubits can be induced with local operations.[1]

[1] Cautrès, C., Mhalla, Perdrix, Savin, Thomassé, Vertex-minor universal graphs for generating entangled quantum subsystems, ICALP 2024

Local equivalences of hypergraph states

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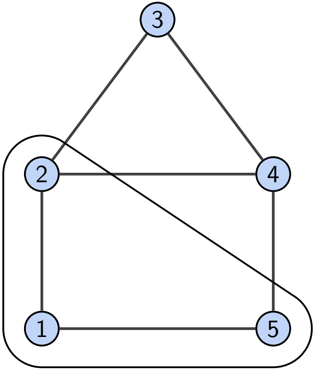

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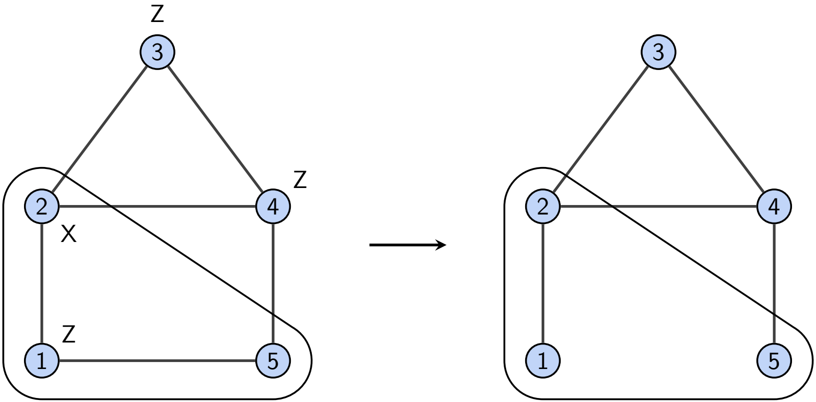

Thanks

arXiv:2409.20183

arXiv:2502.06566