Local equivalence of stabilizer states

a graphical characterisation

Nathan Claudet and Simon Perdrix

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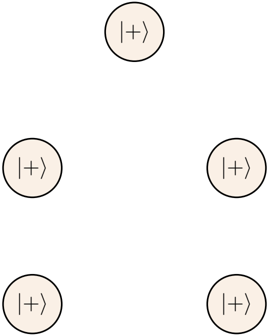
Outline

1. Graph states, local unitary equivalence, local Clifford equivalence & local complementation
2. Generalising local complementation to capture local unitary equivalence
3. An infinite strict hierarchy of local equivalences
4. Conclusion

Graph states, local unitary equivalence, local Clifford equivalence & local complementation

Graph states

A graph state is a quantum state represented by an undirected[1] and simple[2] graph. The vertices represent the qubits and the edges represent entanglement.



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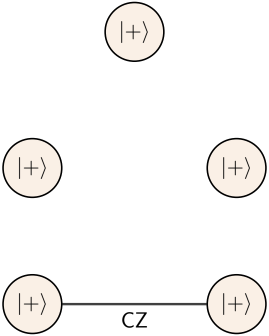

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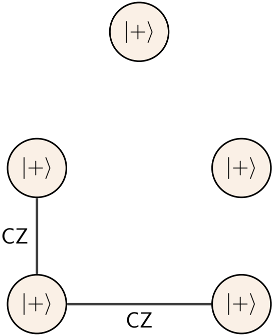

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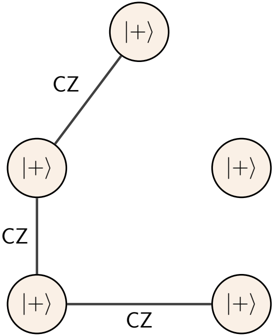

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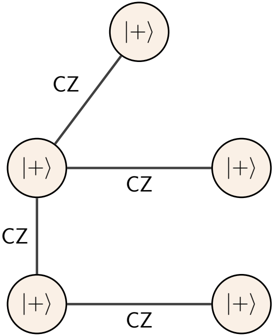

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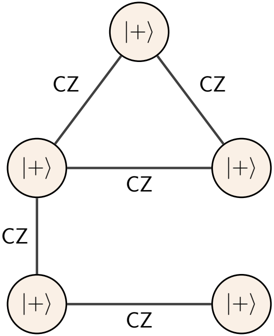

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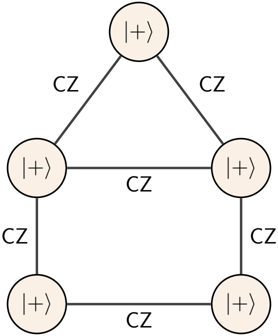

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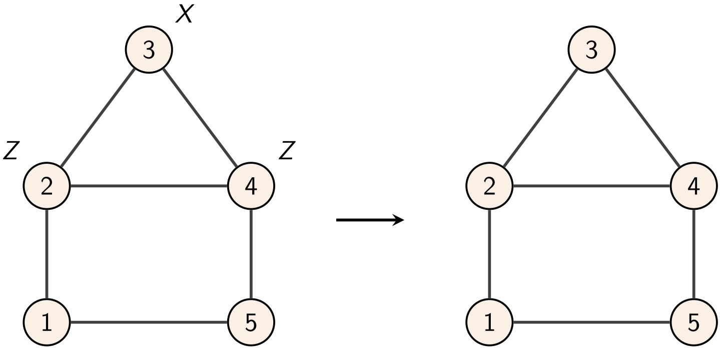
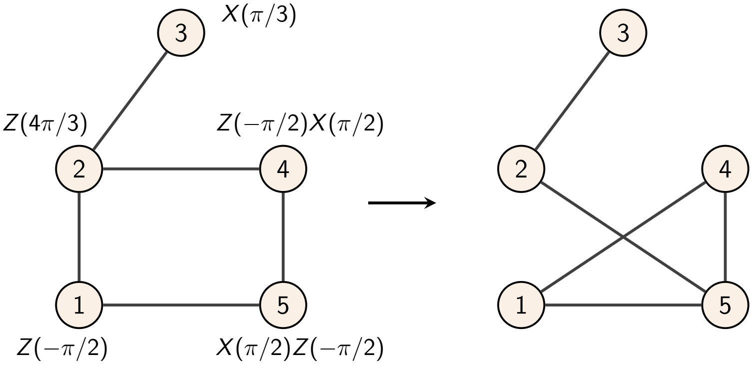
Stabilizer states

Graph states are a subfamilly of stabilizer states because for each vertex u, applying X on u and applying Z on the neighbours of u leaves the graph state invariant.



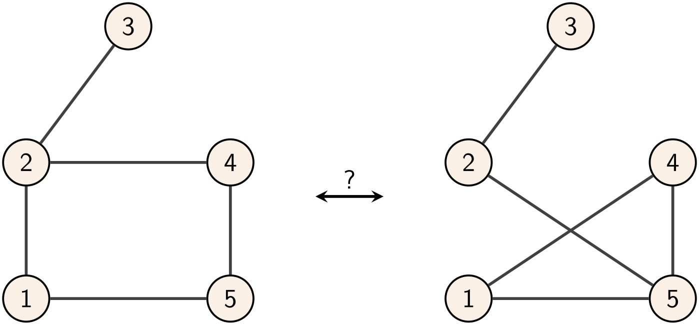
Entanglement of graph states

Two different graph states can have the same entanglement, i.e. they are related by local unitaries. In this case they are **local unitary equivalent** (or LU-equivalent).



Recognising local unitary equivalence

Is there an easy way to recognise whether two graph states are local unitary equivalent ?



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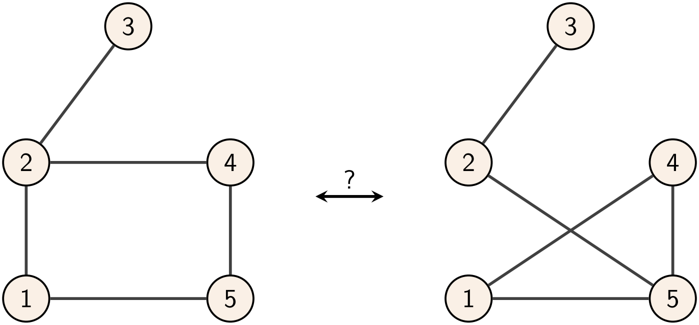
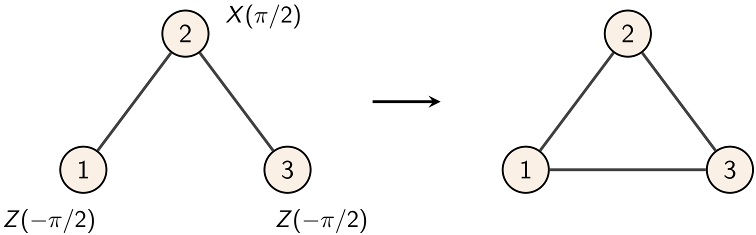


Short answer: **No.**

An easier subproblem: local Clifford equivalence

Two graph states are said **local Clifford equivalent** (or LC-equivalent) if there are related by unitaries in the local Clifford group: $< H, Z(\pi/2) >$



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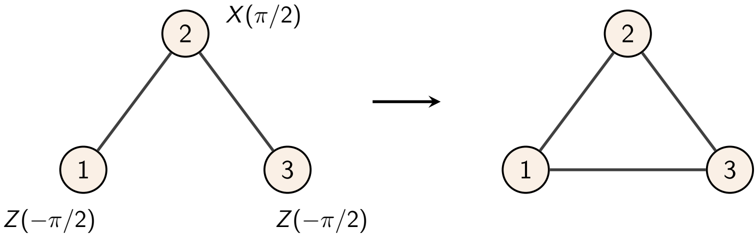


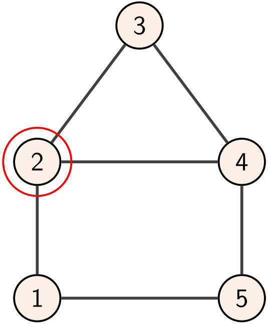
Theorem (Van den Nest, Dehaene, De Moor, 2004)

Two graph states are local Clifford equivalent iff the two corresponding graphs are related by **local complementations***.*

Local complementation

Definition

A local complementation on a vertex u consists in complementing the (open) neighbourhood of u.



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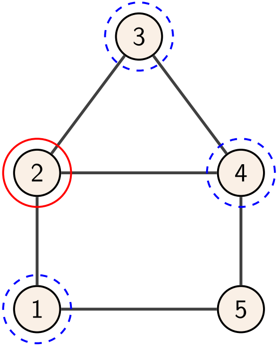

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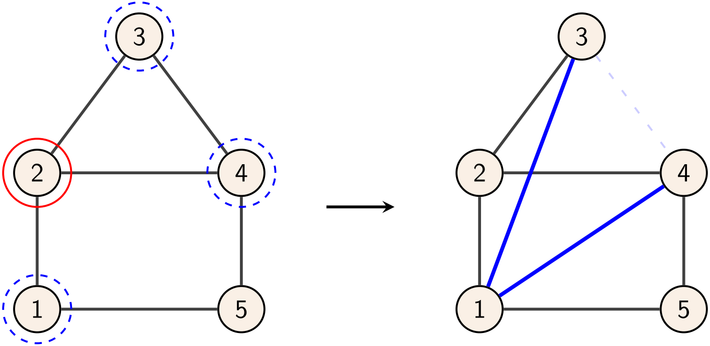

Algorithmic aspect of local Clifford equivalence

There exists an efficient algorithm (Bouchet, 1991) to recognise whether two graphs are related by local complementations, implying an efficient algorithm to recognise whether two graphs are local Clifford equivalent.

The LU=LC conjecture

Conjecture

Two graph states are local unitary equivalent iff they are local Clifford equivalent.

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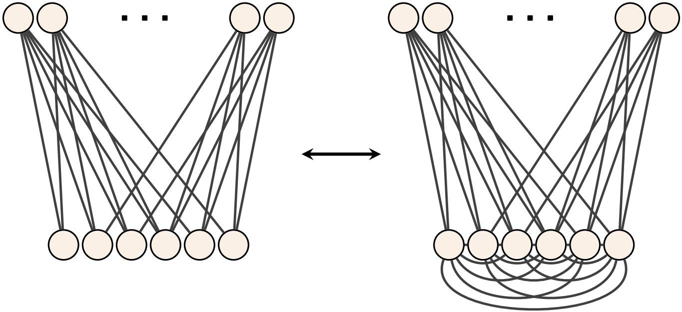
False ! a 27-vertex counter-example (Ji et al., 2008) :



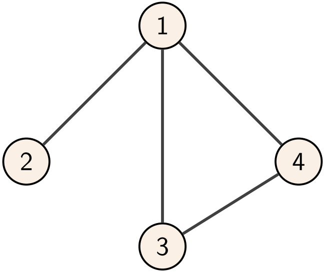
Consequence: local complementation does **not** capture the local unitary equivalence of graph states.

Generalising local complementation to capture local unitary equivalence

Adding weight to edges

Let's write graphs as **weighted graphs**.

- Edges become edges of weight 1.
- Non-edges become edges of weight 0.



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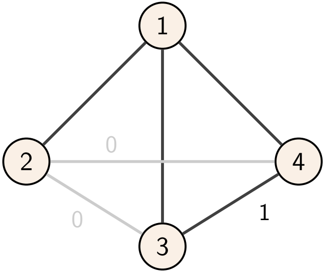
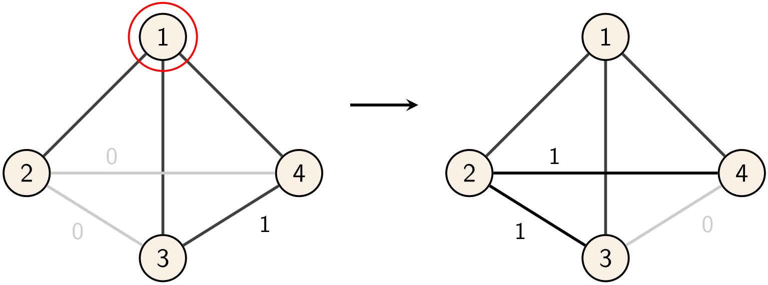


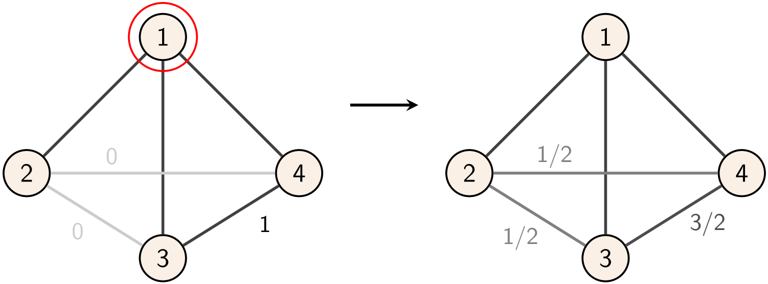
Local complementation on a weighted graph

In the formalism of weighted graphs, a local complementation over a vertex u consists in adding **mod 2** a weight 1 to edges between the neighbours of *u*.



Towards a finer local complementation

One way to define a finer local complementation is to add (mod 2) a weight $1/2$.



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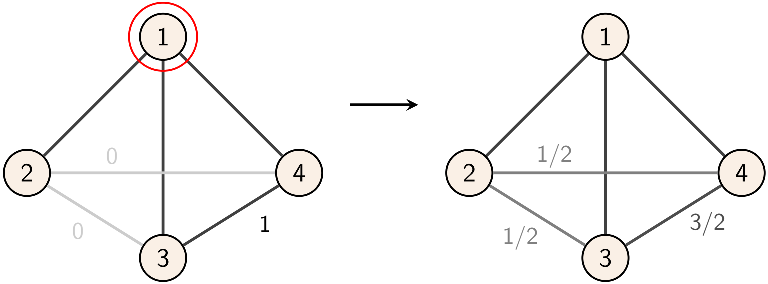
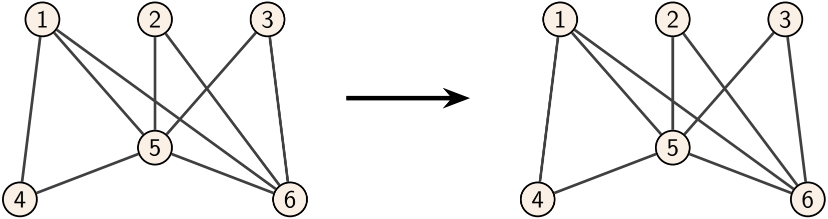


Problem: The outcome is not a graph !

Multiple local complementations at once

Remark: we may define local complementation on an independent set of vertices[3] as the local complementations commute.



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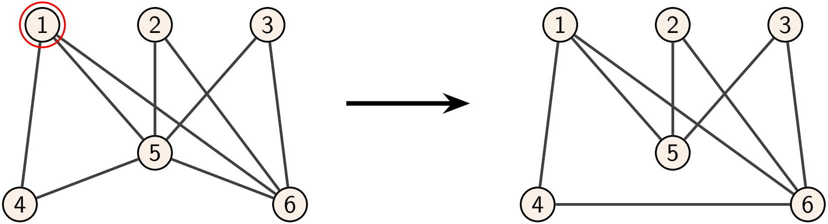

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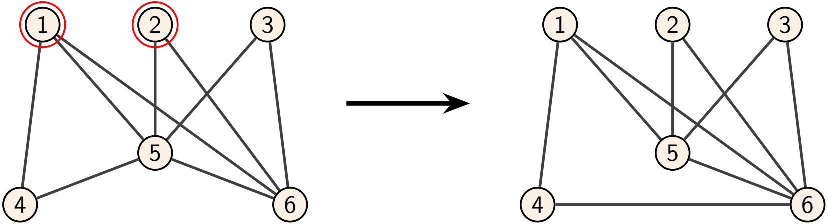

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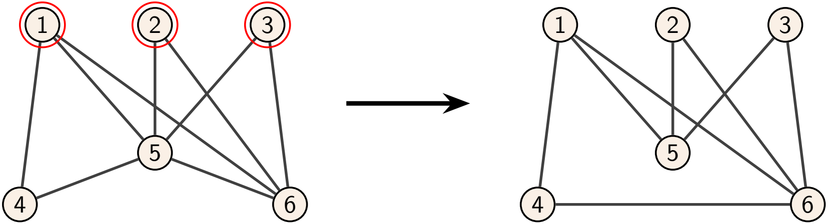

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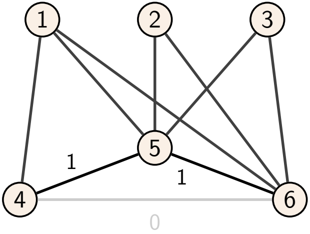
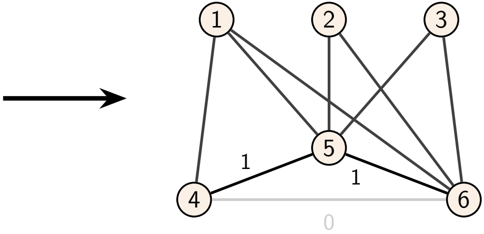
Now, with multiple finer local complementation adding (mod 2) weights that are multiples of $1/2$, the outcome may be a graph.





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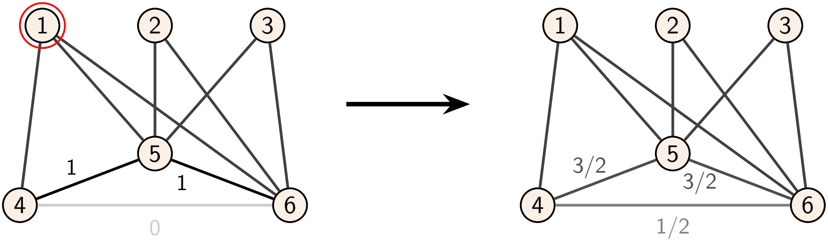

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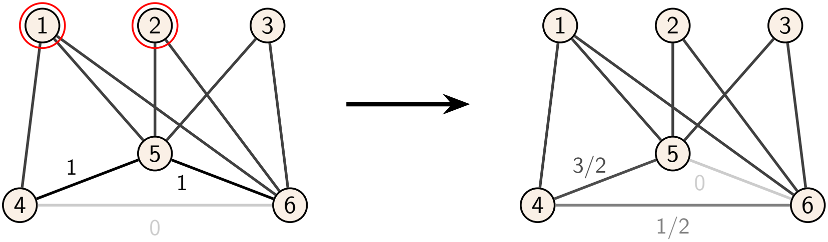

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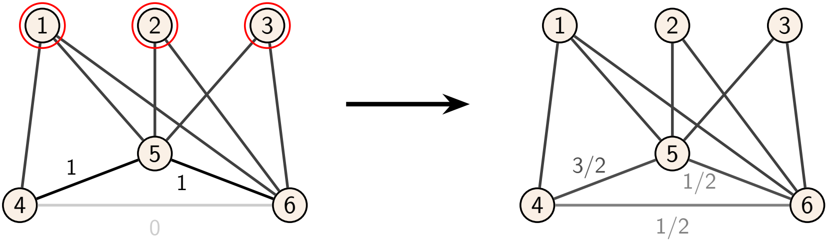

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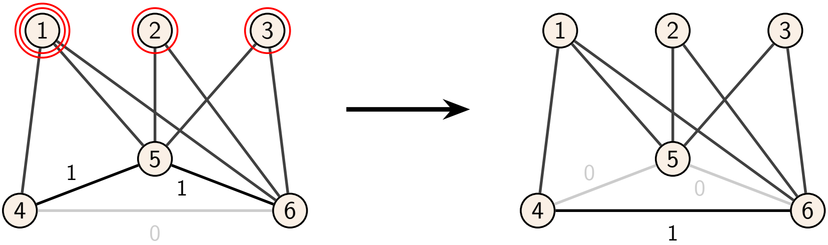

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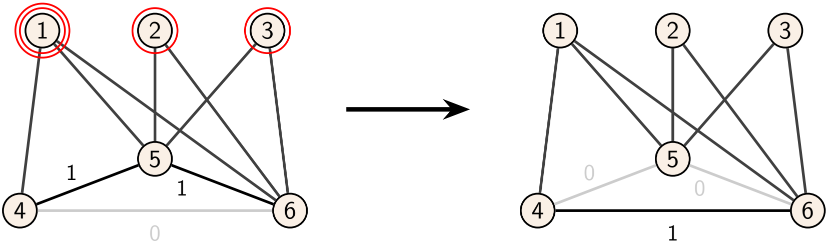

When the outcome is a graph (i.e. all weights are 0 or 1 mod 2), this operation is well defined (with some additional conditions). We call it a **2-local complementation**.

r-local complementation

We define an infinite family of similar operations parametrised by an integer r:

r-local complementations

In an r-local complementation we add (mod 2) weights that are multiples of $1/2^r$.

r-local complementation captures a generalisation of local Clifford equivalence

Theorem (Van den Nest, Dehaene, De Moor, 2004)

Two graph states are related by local unitaries in $< H, Z(\pi/2) >$ *(Clifford) iff the two corresponding graphs are related by local complementations.*

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Two graph states are local unitary equivalent iff the two corresponding graphs are related by r-local complementations for some r.

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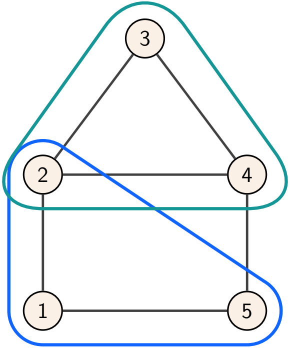
Corollary

If two graph are local unitary equivalent, the local unitaries can be chosen to be in the dyadic fragment generated by H and $Z(\pi/2^r)$ for any r.

Proof sketch: Minimal local set

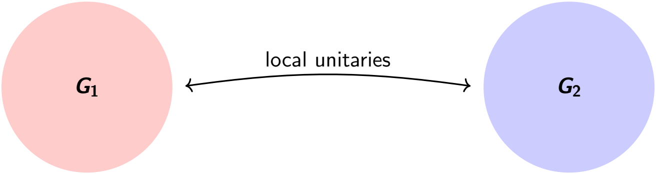


Minimal local sets are subsets of vertices that carry information on the possible local unitaries that maps graph states to other graph states.

Theorem (Claudet, Perdrix, 2024)

Each vertex of a graph is covered by at least one minimal local set.

Proof sketch: Standard form for graph states



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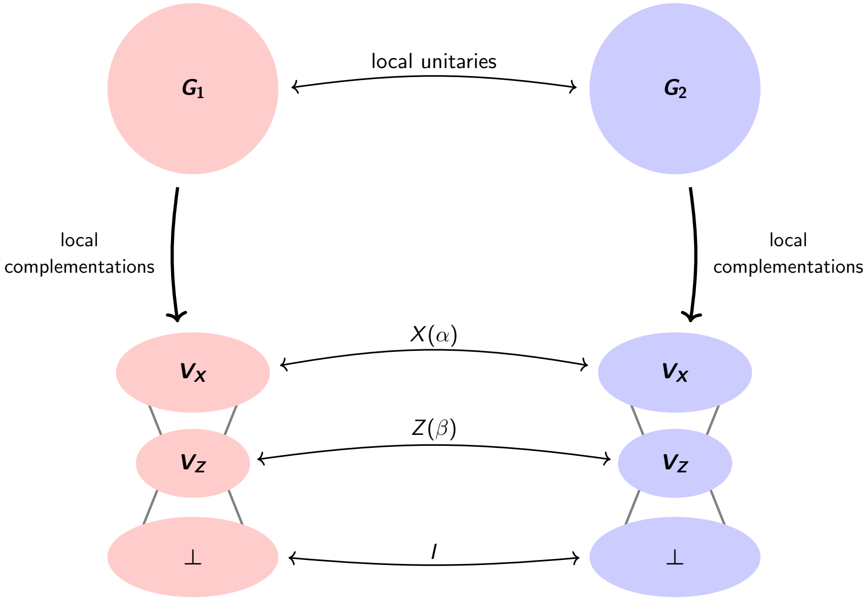

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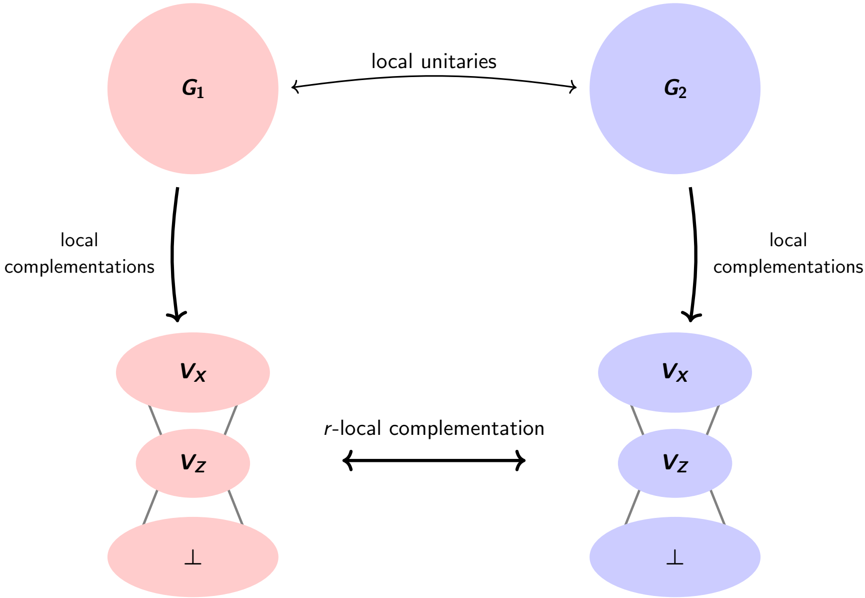

Application: local equivalence of repeater graph states

For some families of graph states, LU=LC.

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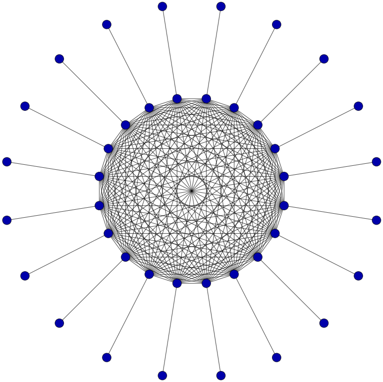
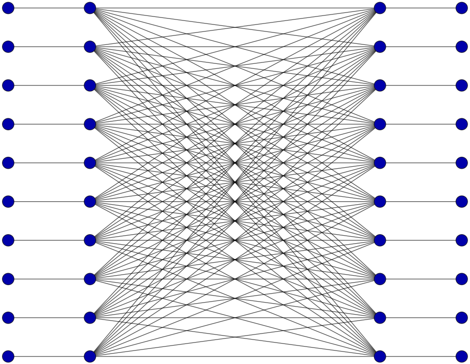
Example: some repeater graph states (Tzitrin, 2018)





An infinite strict hierarchy of local equivalences

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Natural question: Is 2-local complementation sufficient to characterise local unitary equivalence ? 3-local complementation ?

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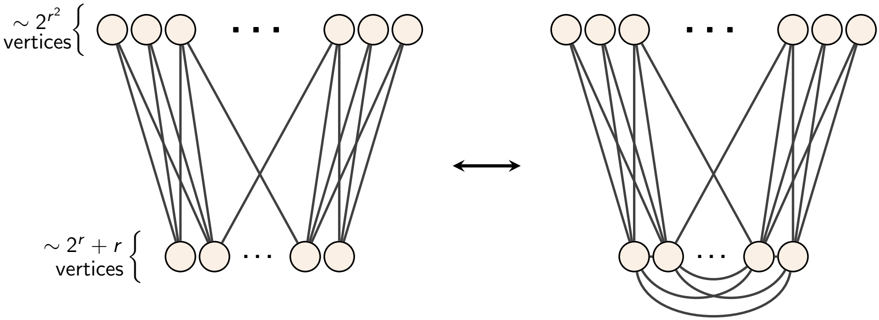
Natural question: Is 2-local complementation sufficient to characterise local unitary equivalence ? 3-local complementation ?

Answer: No, r-local complementation for every r is necessary to characterise local unitary equivalence.

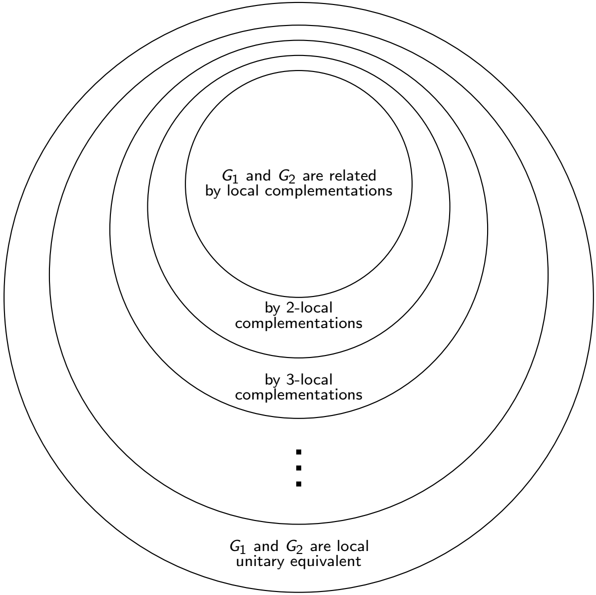
Constructive proof

For any r, we construct a pair of graphs that are related by $(r+1)$-local complementations but not r-local complementations.



Every upper vertex is related to exactly $2^r + 1$ vertices.

The infinite strict hierarchy



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- Algorithmic aspect of local unitary equivalence

Thanks

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