k-vertex-minor universal graphs

and small k-pairable quantum states

Nathan Claudet, Mehdi Mhalla, Simon Perdrix

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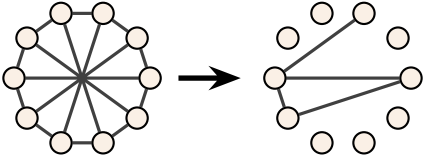
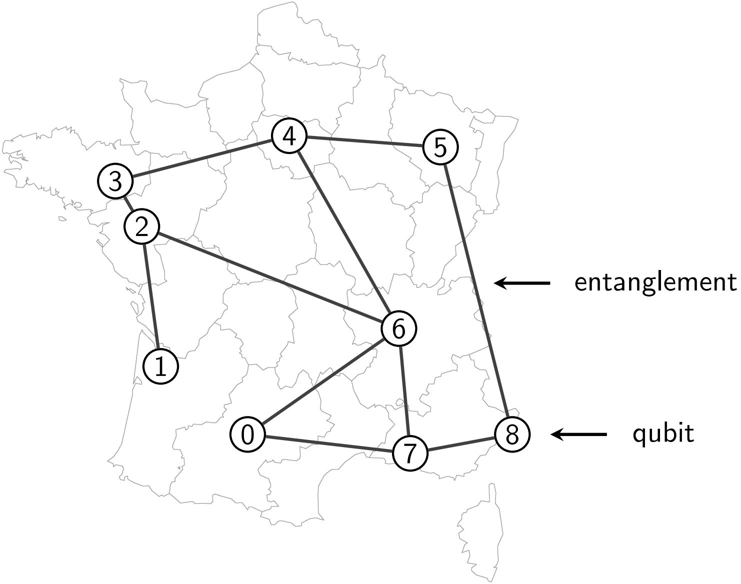


Outline

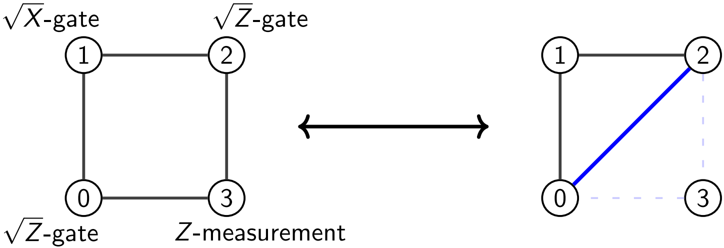
Motivation - quantum communication networks

Quantum communication networks



Correspondence between quantum graph states and graphs

$|G\rangle$ G



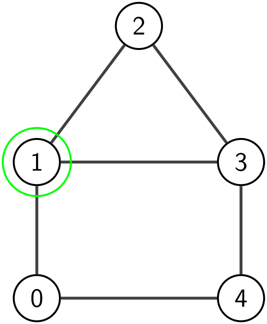
local (i.e. single-qubit)
quantum operations

local complementations
& vertex disconnections

Local complementation

Definition (Local complementation)

Given a graph G, a local complementation on a vertex u consists in complementing the (open) neighborhood of u in G.



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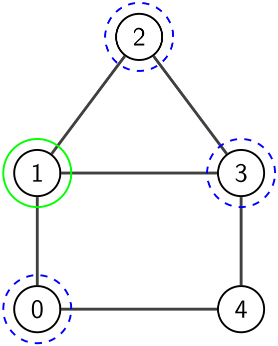

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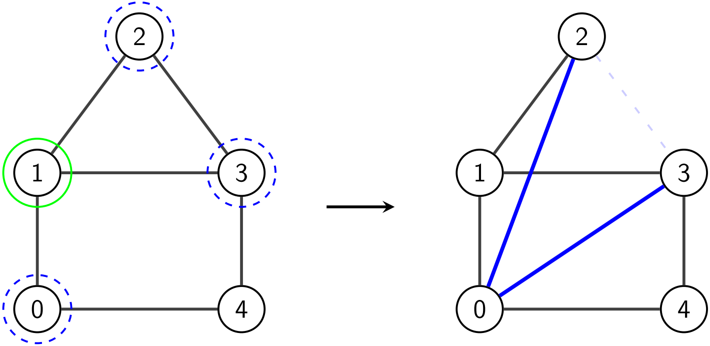
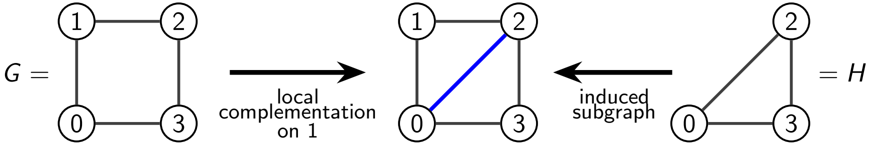


k-vertex-minor universal graphs

Definition (Vertex-minor)

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ such that $V_H \subseteq V_G$, H is a vertex-minor of G if H can be obtained as a induced subgraph of G by means of local complementations.



Vertex-minors

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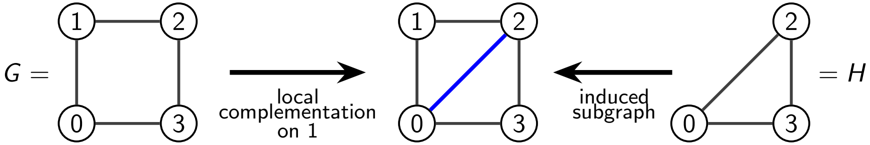
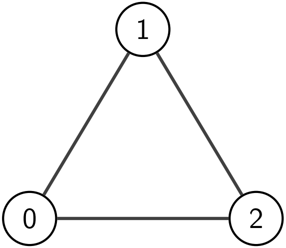


Definition

A graph G is *k-vertex-minor universal* if any graph on any k vertices is a vertex-minor of G.

k-vertex-minor universal graphs : example 1

K_3 is 2-vertex-minor universal.



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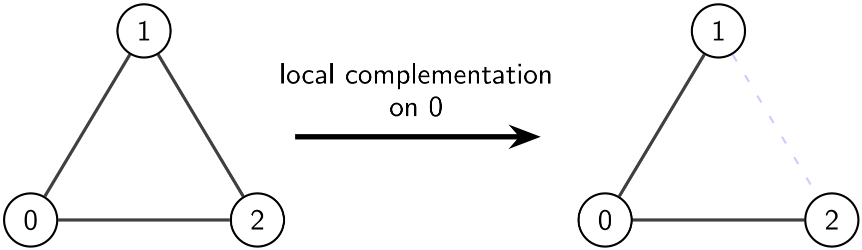

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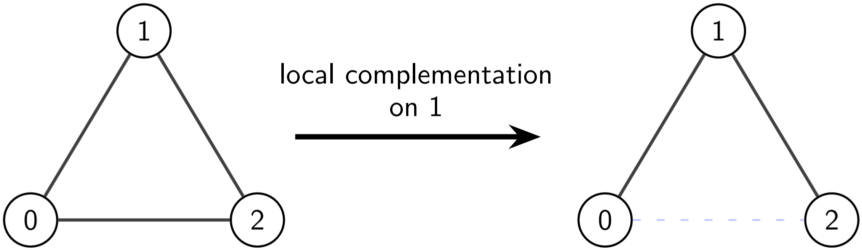

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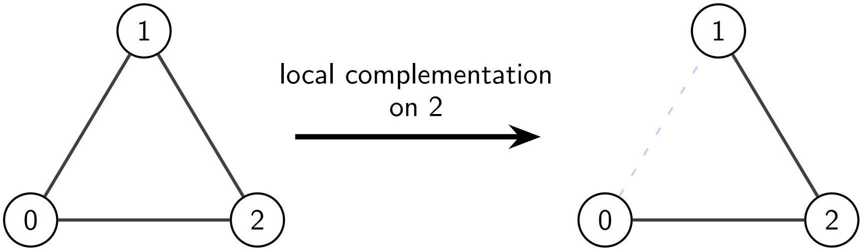
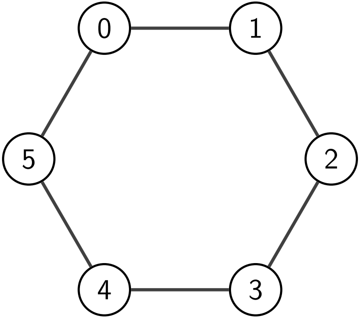


k-vertex-minors-universal graphs : example 2

C_6 is 3-vertex-minor universal.



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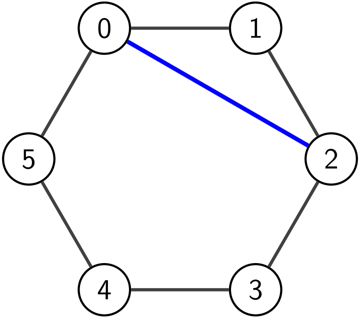

To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.

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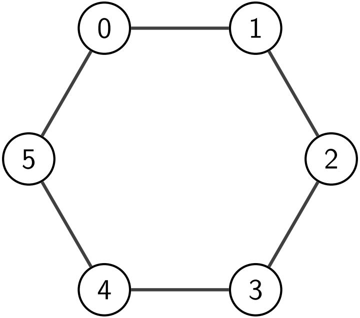

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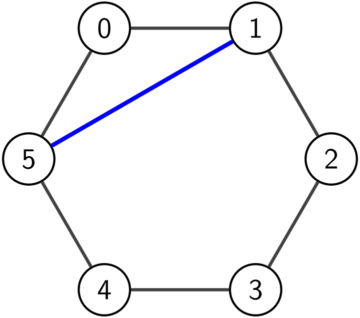

To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
To induce the empty graph on $\{0, 1, 2\}$: Local complementation on 0,

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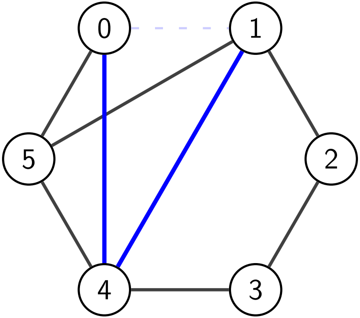

To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
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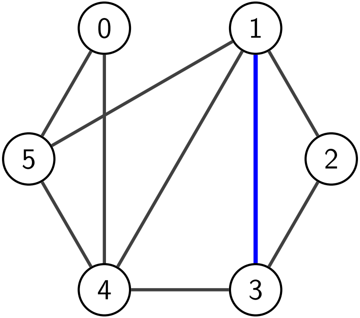

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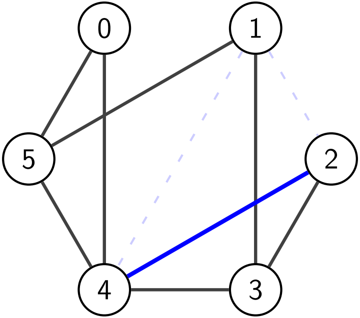

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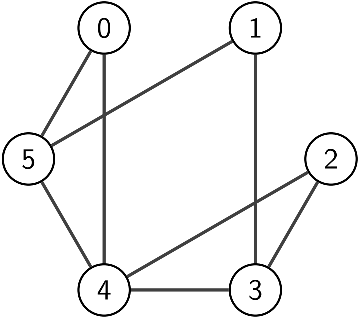

To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
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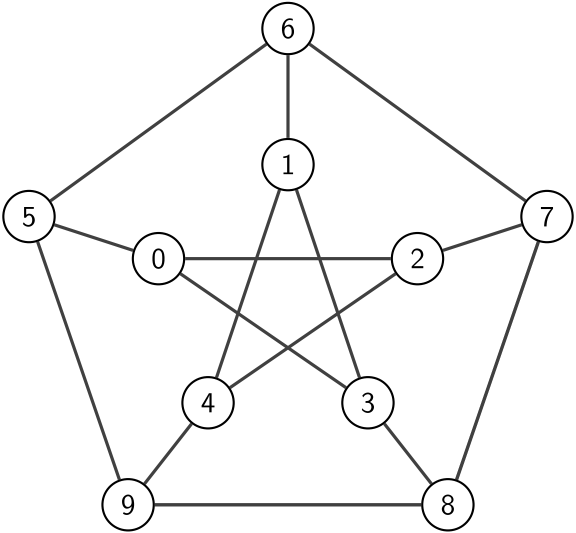
k-pairable states : example 3

The Petersen graph is 4-vertex-minor universal.



Some results

Polynomial-size k-vertex-minors-universal graphs

Proposition (Claudet,Mhalla,Perdrix 2023)

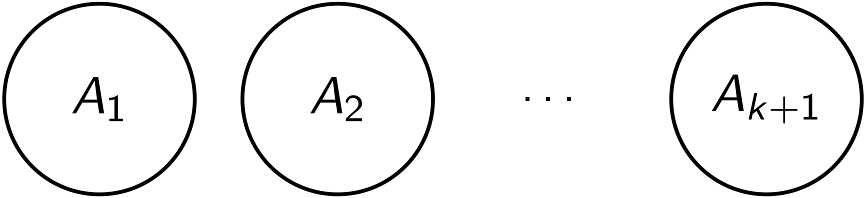
For any k, there exists a k-vertex-minor universal graph of order $n = O(k^4 \ln(k))$.

Polynomial-size k-vertex-minors-universal graphs

Proposition (Claudet,Mhalla,Perdrix 2023)

For any k, there exists a k-vertex-minor universal graph of order $n = O(k^4 \ln(k))$.

Idea of proof: Probabilistic construction of a balanced $(k+1)$-partite graph.



Probability of two vertices u and v sharing an edge : 0 if u and v are in the same component, $2/k$ else. $\forall K \in \binom{V}{k}$, $\exists i$ s.t. $K \cap A_i = \emptyset$.

An upper bound on k-vertex-minor universality

Proposition

For any k, a k-vertex-minor universal graph is at least of order $n = O(k^2)$.

A variation : k-pairable graphs

Definition

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Proposition

$2k$-vertex-minor universality $\implies$ k-pairability.

Proposition (Claudet,Mhalla,Perdrix 2023)

For any k, there exists a k-pairable graph on $n = O(k^3 \ln^3(k))$ qubits.

Conclusion

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Introduction of the notion of k-vertex-minor universal graphs.
Main result: for any k there exist k-vertex-minor universal graph of order $O(k^4 \ln(k))$.

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Future work:

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Future work:

- Explicit constructions?
- Existence of k-vertex-minor universal graphs of order $O(k^3)$? $O(k^2)$?

Conclusion

Introduction of the notion of k-vertex-minor universal graphs.
Main result: for any k there exist k-vertex-minor universal graph of order $O(k^4 \ln(k))$.

Future work:

- Explicit constructions?
- Existence of k-vertex-minor universal graphs of order $O(k^3)$? $O(k^2)$?
- k-pairability $= 2k$-vertex-minor universality?

Thanks

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