Recognizing graph states with the same entanglement by looking at the underlying graphs

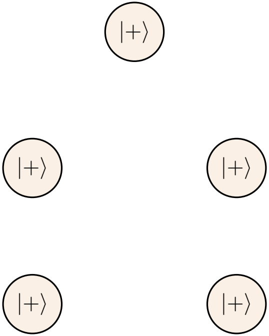
Nathan Claudet

University of Innsbruck - 07/04/25

Graph states, local unitary equivalence, local Clifford equivalence & local complementation

Graph states

A graph state is a quantum state represented by an undirected[1] and simple[2] graph. The vertices represent the qubits and the edges represent entanglement.



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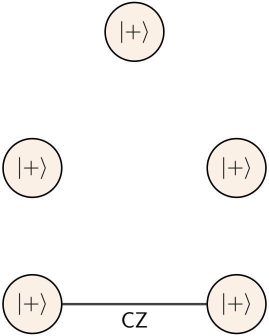

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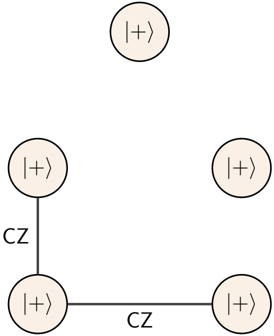

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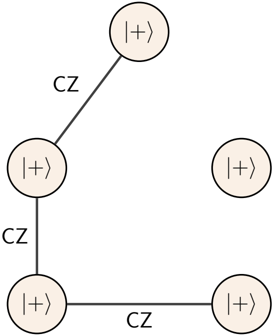

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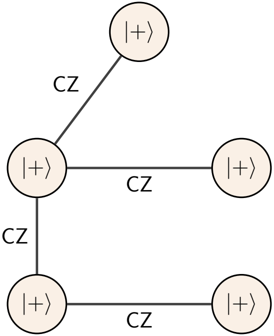

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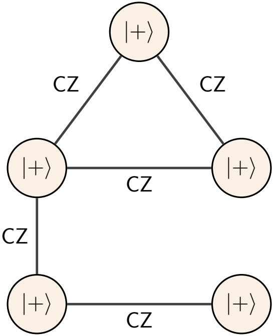

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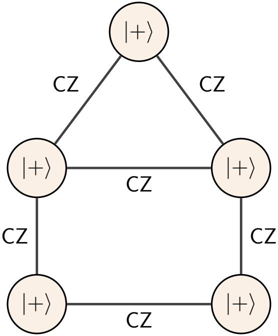

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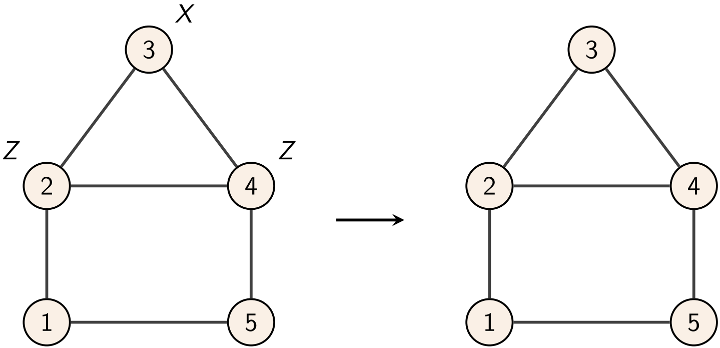
Stabilizer states

Graph states are a subfamilly of stabilizer states because for each vertex u, applying X on u and applying Z on the neighbours of u leaves the graph state invariant.



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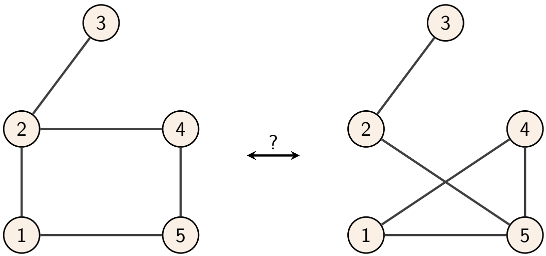

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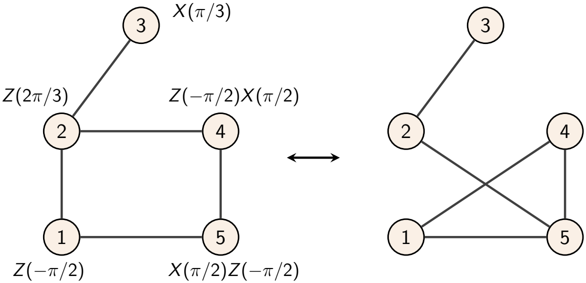
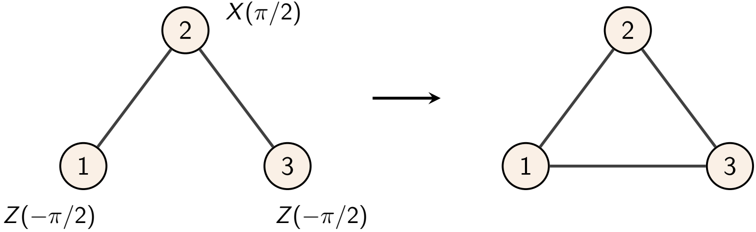


An easier subproblem: local Clifford equivalence

Two graph states are said **local Clifford equivalent** (or LC-equivalent) if they are related by unitaries in the local Clifford group.
Single-qubit Clifford group $= < H, Z(\pi/2) >$.



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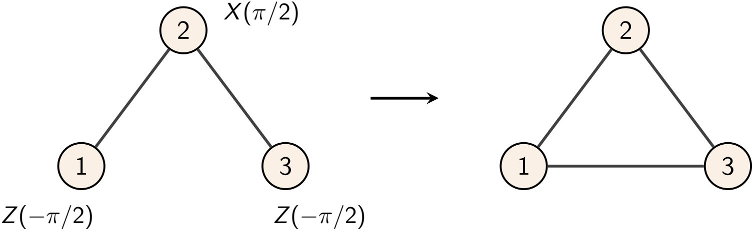


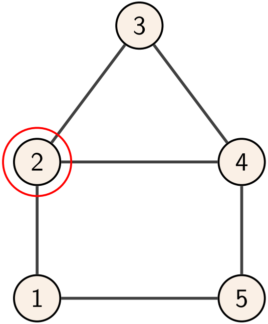
Theorem (Van den Nest, Dehaene, De Moor, 2004)

Two graph states are LC-equivalent iff the two corresponding graphs are related by **local complementations***.*

Local complementation

Definition (Kotzig, 1966)

A local complementation on a vertex u consists in complementing the (open) neighbourhood of u.



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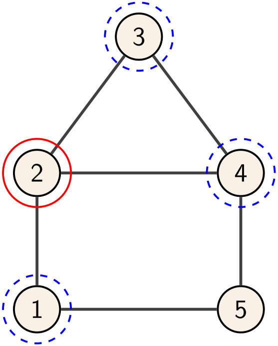

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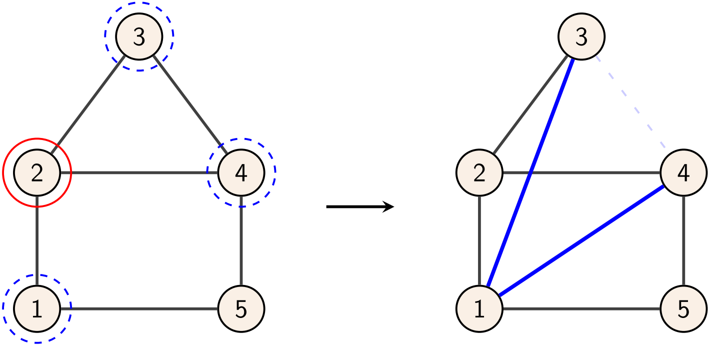

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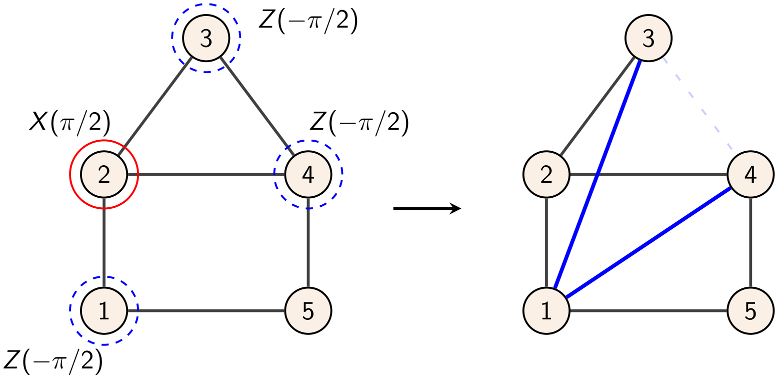

Algorithmic aspect of local Clifford equivalence

Proposition (Bouchet, 1991)

There exists an efficient algorithm to decide if two graphs are related by local complementations.

$\rightarrow$ algorithm to decide if two graph states are LC-equivalent.

Quick history of the LU=LC conjecture

The LU=LC conjecture

Formulated in the early 2000's.

Conjecture

LU=LC i.e. if two graph states are LU-equivalent (local unitaries) then they are LC-equivalent (local Clifford).

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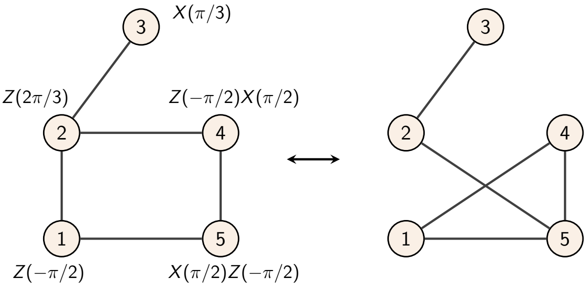

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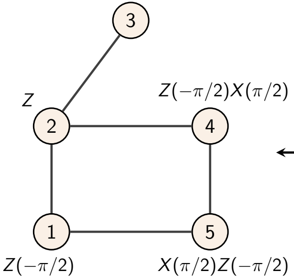

Preliminary results towards LU=LC

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Proposition (Gross, Van den Nest, 2007)

If two graph states are LU-equivalent, the local unitaries can be chosen to be of the form $C_1 Z(\theta) C_2$ where C_1 and C_2 are Clifford.

LU $\neq$ LC

However, LU $\neq$ LC, i.e. local unitary equivalence and local Clifford equivalence do not coincide.

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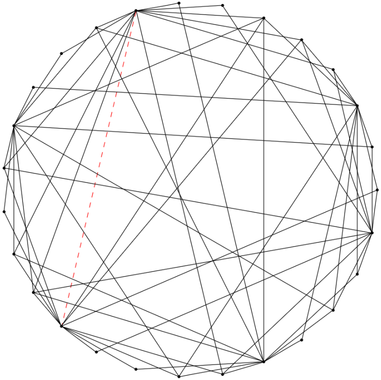
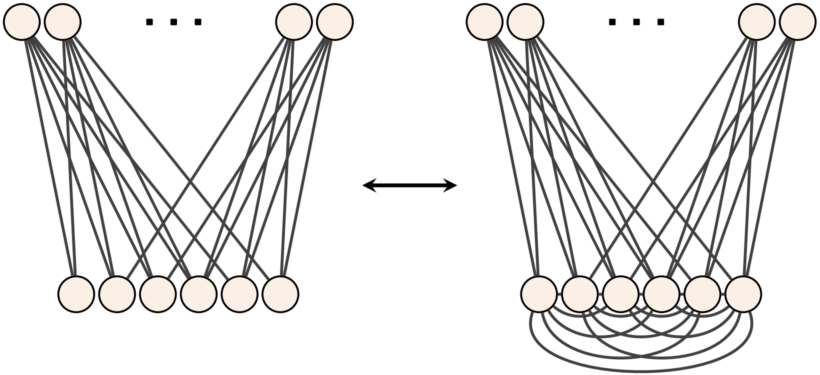
However, LU $\neq$ LC, i.e. local unitary equivalence and local Clifford equivalence do not coincide. $\rightarrow$ 27-qubit pair of graph states that are LU-equivalent but not LC-equivalent (Ji et al. 2008).



Another look at the 27-vertex counterexample

The 27-vertex counterexample is LC-equivalent to a prettier pair of graphs (Tsimakuridze, Gühne, 2017).



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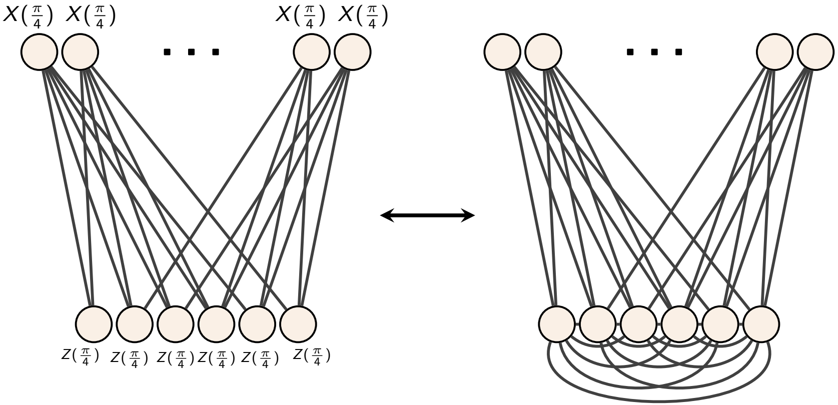

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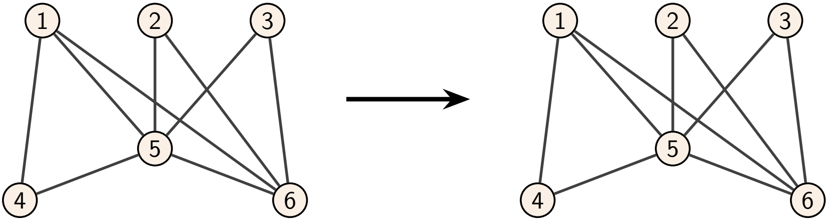
But what about LU-equivalence for **any** graph ? Can we construct a graphical characterisation ?

Generalizing local complementation to capture local unitary equivalence

A refinement of idempotent local complementations

A sequence of local complementations may leave the graph invariant.



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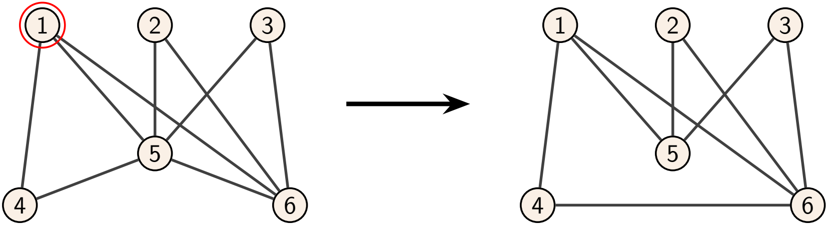

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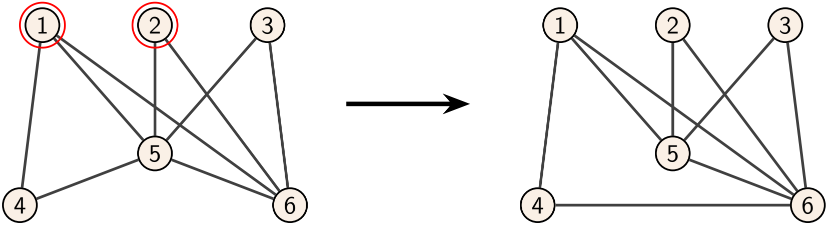

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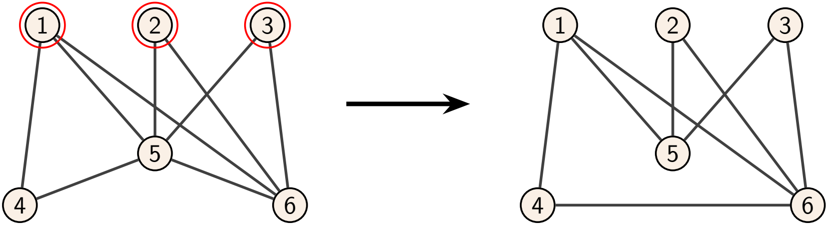

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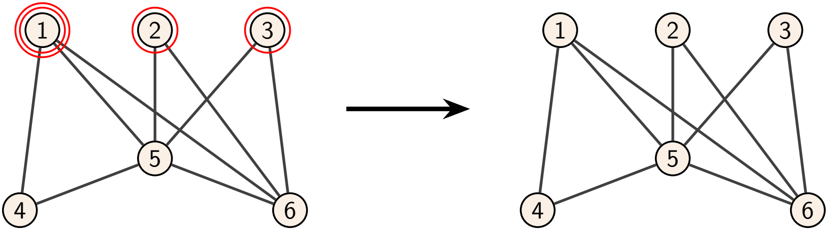

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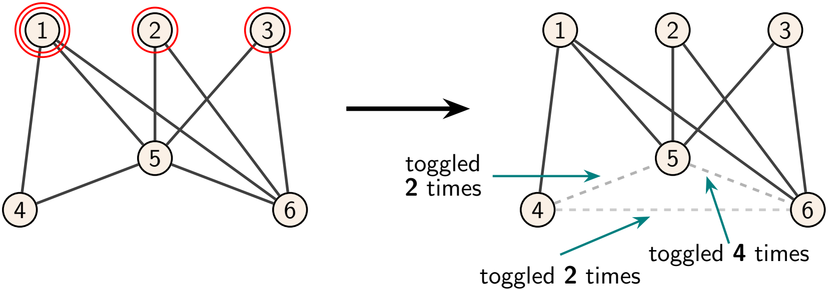

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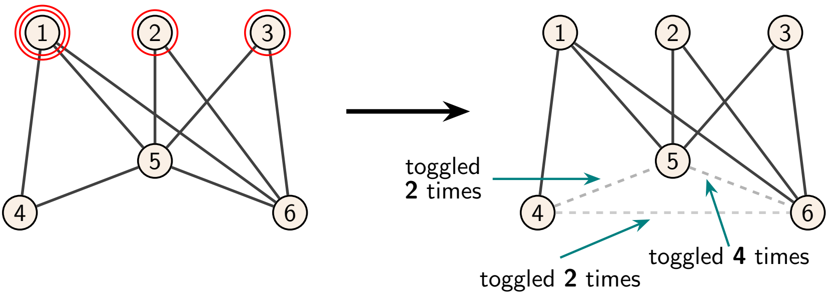

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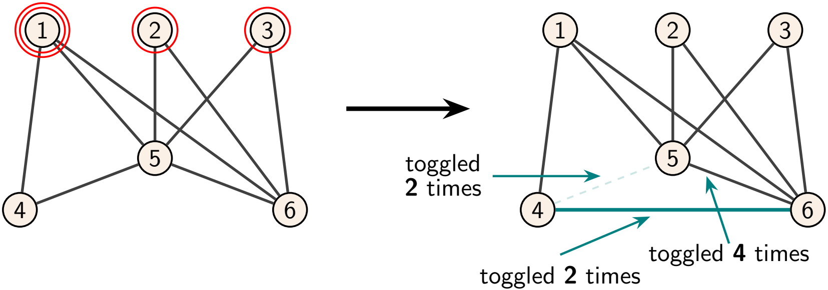

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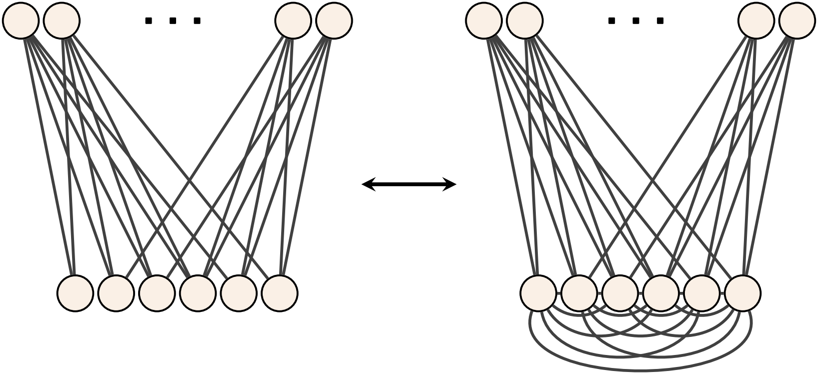

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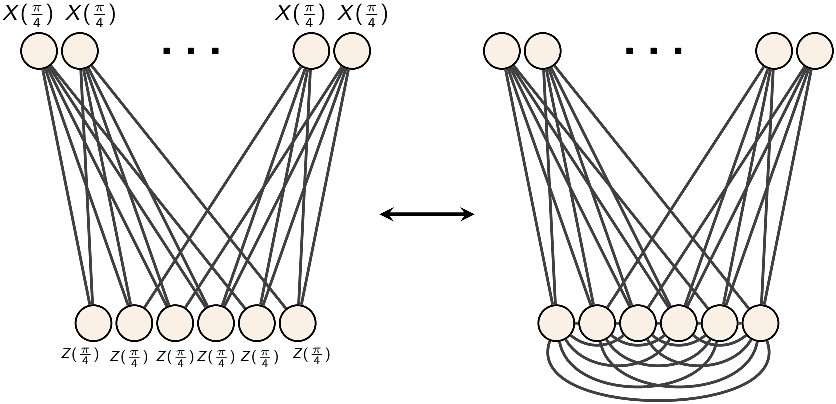

r-local complementation

3-local complementation is a refinement of idempotent 2-local complementation, and so on...
$\rightarrow$ Infinite family of graphical operations parametrised by an integer r:

r-**local complementations**

1-local complementation = local complementation.

Graphical characterization of entanglement

Recall: LC-equivalent $\Leftrightarrow$ related by local unitaries in $< H, Z(\pi/2) >$.
Define: **LC$_r$-equivalent** $\Leftrightarrow$ related by local unitaries in $< H, Z(\pi/2^r) >$.

Graphical characterization of entanglement

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Two graph states are LC$_r$-equivalent iff the two corresponding graphs are related by r-local complementations.

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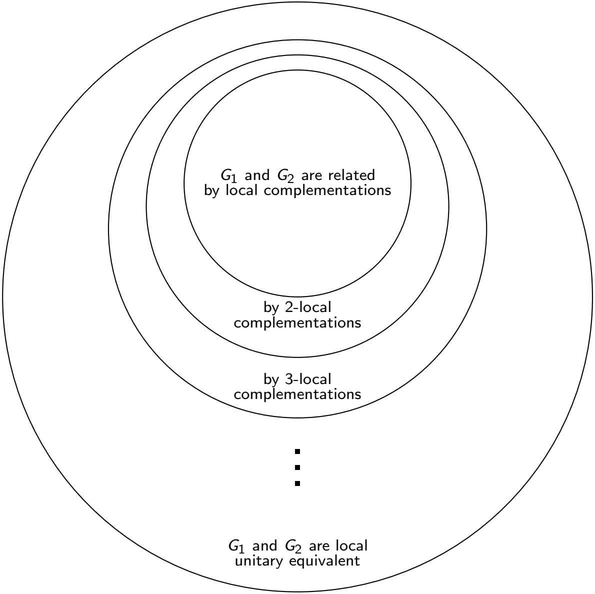
An infinite hierarchy of local equivalences



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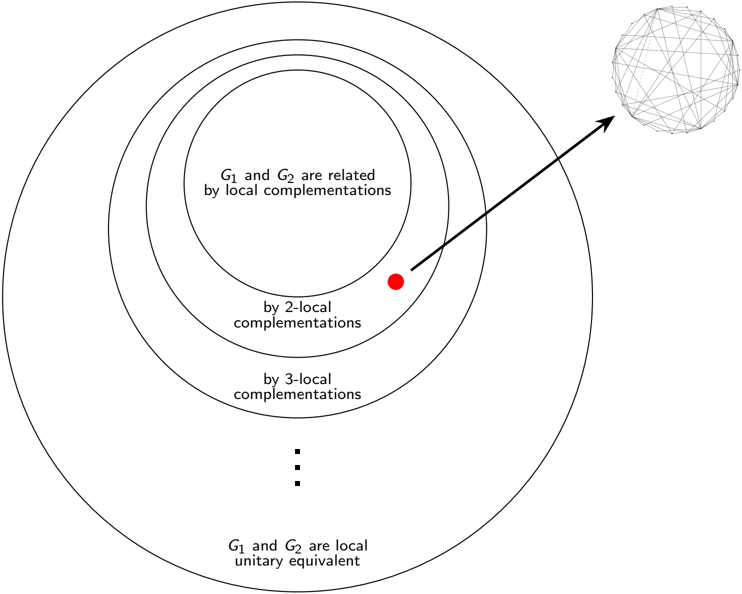

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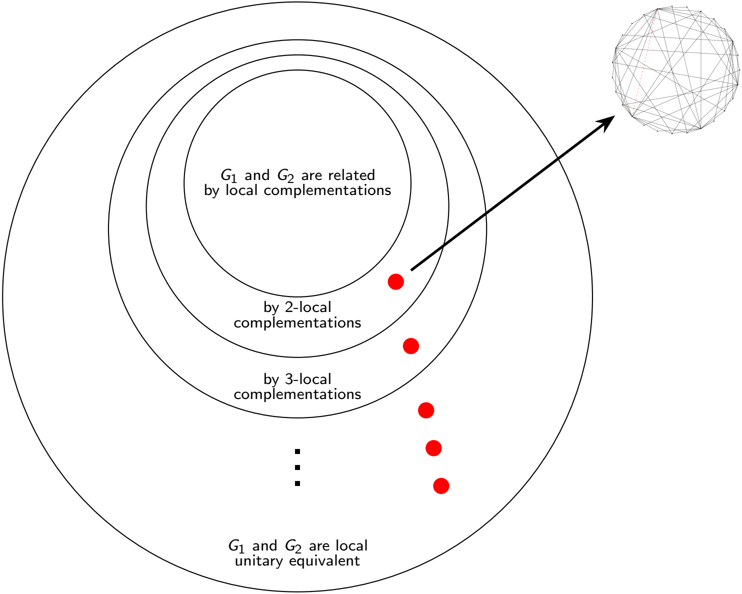
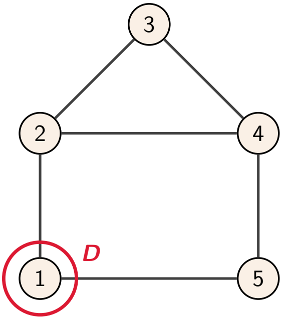


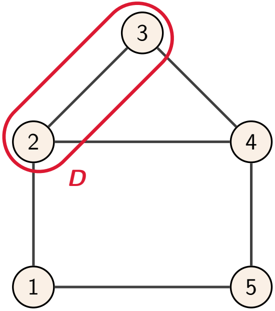
Proof that r-local complementation captures LU-equivalence

Minimal local sets

Definition (Odd neighbourhood)

Given a set of vertices D, the **odd neighbourhood** $Odd_G(D)$ of D is the set of vertices that are neighbours of an odd number of vertices in D.





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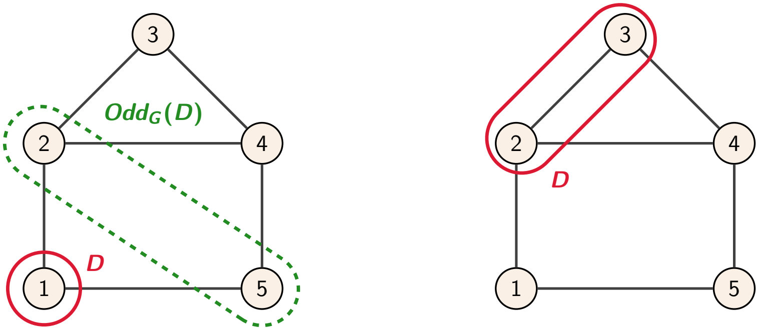

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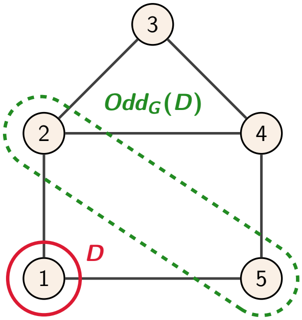
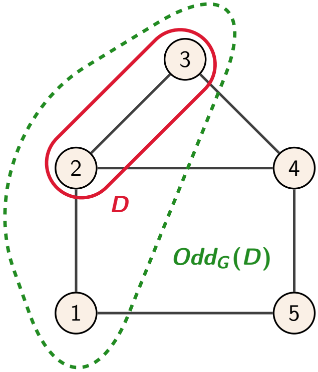
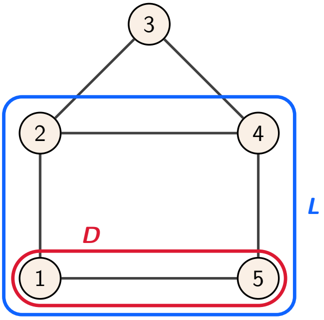




Minimal local sets

Definition

A **local set** is a non-empty vertex set of the form $L = D \cup Odd_G(D)$.
A **minimal local set** is a local set that is minimal by inclusion (i.e it doesn't strictly contain another local set).



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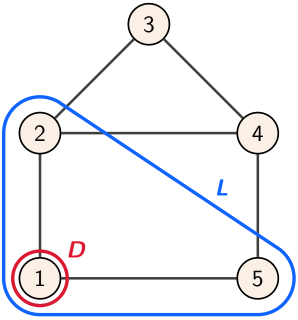

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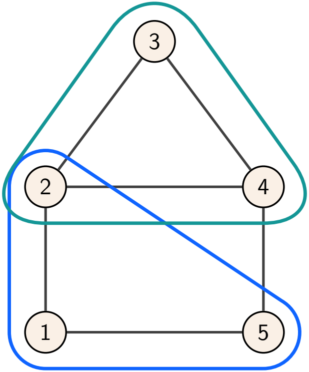
Minimal local sets carry information on the possible local unitaries that maps graph states to other graph states.

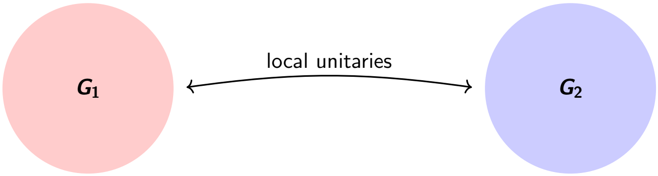
Minimal local sets cover any graph

Theorem (C, Perdrix, 2024)

Each vertex of a graph is covered by at least one minimal local set.



Proof sketch: Standard form for graph states



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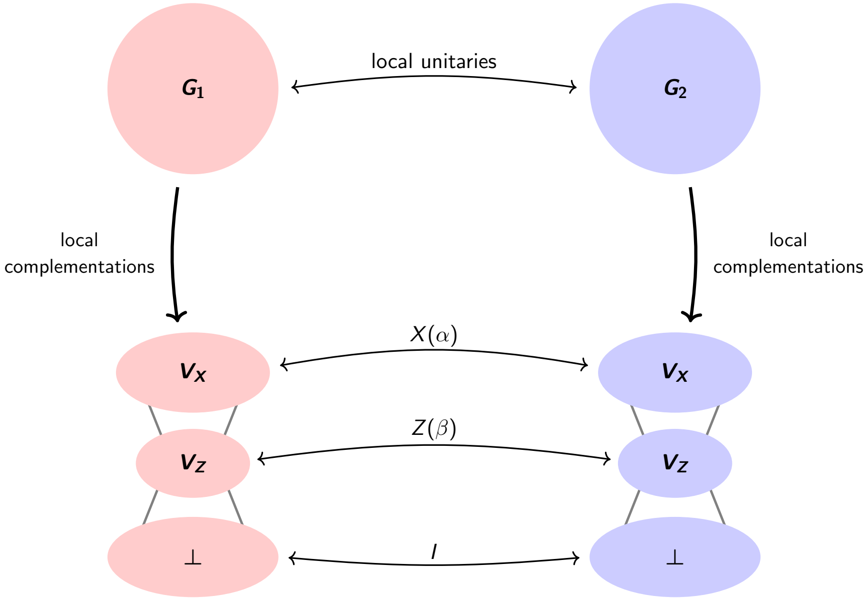

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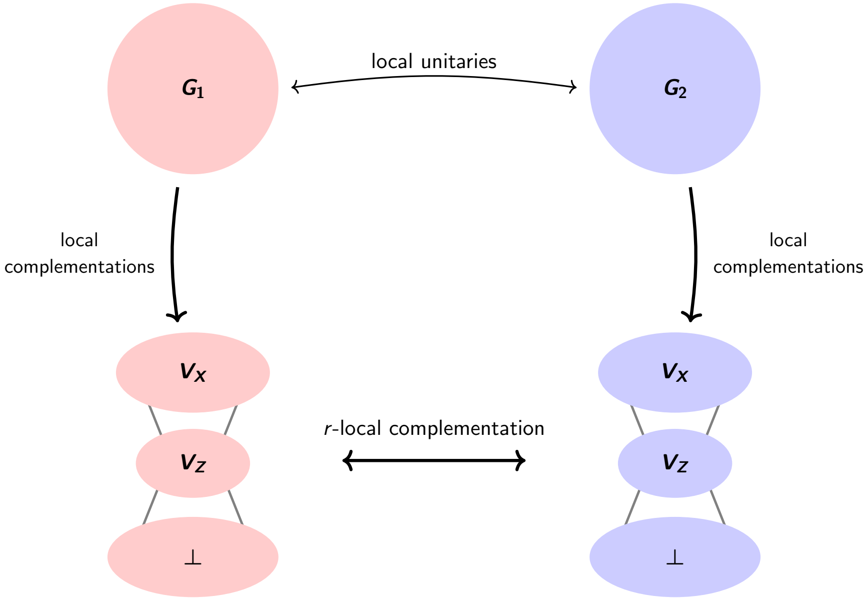

Application 1: a toolbox to prove LU=LC for classes of graphs

An exponential bound on LC$_r$-equivalence

Proposition (<u>C</u>, Perdrix, 2025)

Suppose $n \leqslant 2^{r+3} - 1$. If two graph states over n qubits are LU-equivalent then they are LC$_r$-equivalent i.e. LU=LC$_r$.

An exponential bound on LC_r-equivalence

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Suppose $n \leqslant 2^{r+3} - 1$. If two graph states over n qubits are LU-equivalent then they are LC_r-equivalent i.e. $LU{=}LC_r$.

In particular:

- If $n \leqslant 15$, LU=LC;

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Proposition (C, Perdrix, 2025)

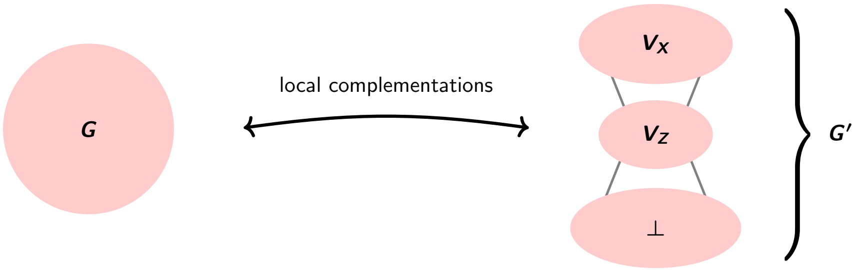
LU=LC for graph states up to 19 qubits.

A graphical characterization for LU=LC

Proposition (C, Perdrix, 2024)

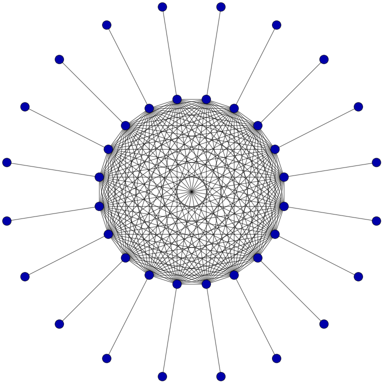
LU=LC for $|G\rangle$ if there exists $|G'\rangle$ such that:

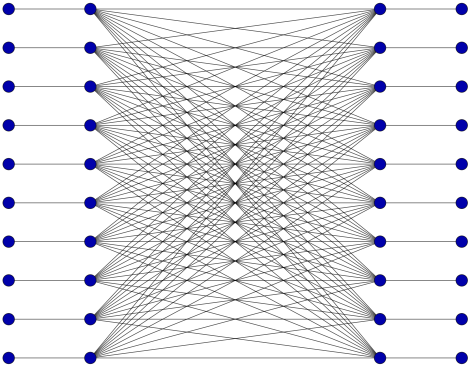
- *$|G\rangle$ and $|G'\rangle$ are LC-equivalent;*
- *G' is in standard form;*
- *Any r-local complementation over V_X can be implemented by usual local complementations.*



Example: LU=LC for repeater graph states

It was conjectured that LU=LC holds for some repeater graph states (Tzitrin, 2018). We showed that this is indeed the case.





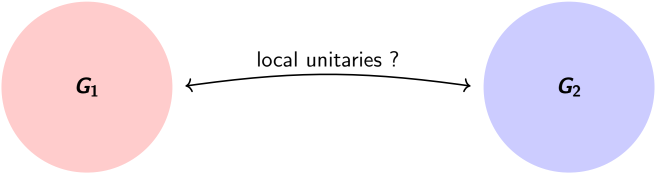
Application 2: a quasi-polynomial algorithm for LU-equivalence

Application 3: A quasi-polynomial algorithm to decide local unitary equivalence

Theorem (C, Perdrix, 2025)

There exists an algorithm that decides if two graph states are LU-equivalent with runtime $n^{log_2(n)+O(1)}$.

Step 1: Standard form



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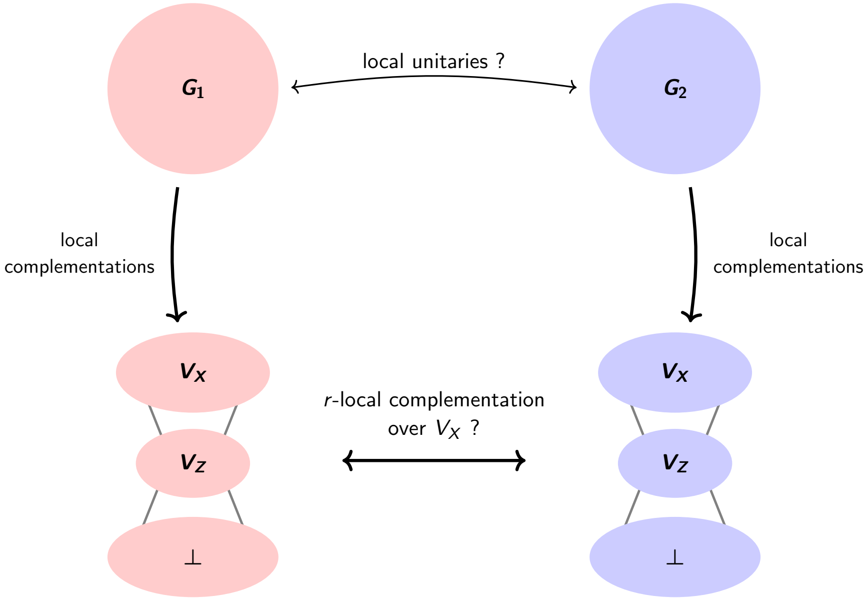
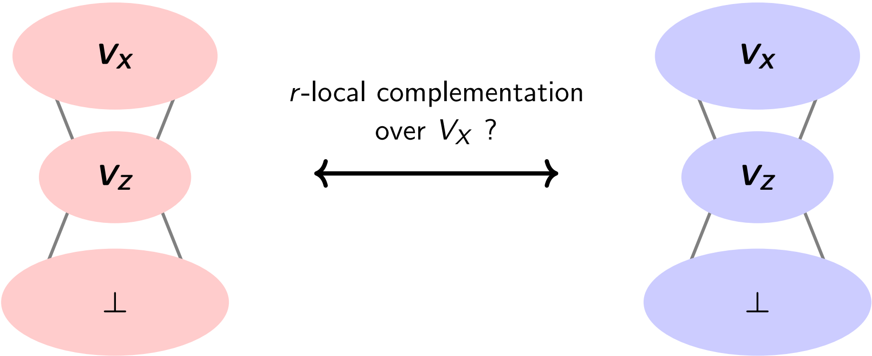


Step 2: Reduction to LC-equivalence with constraints



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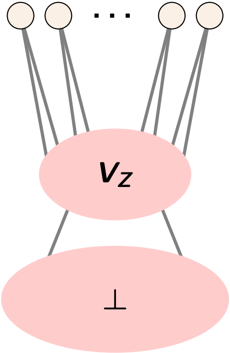
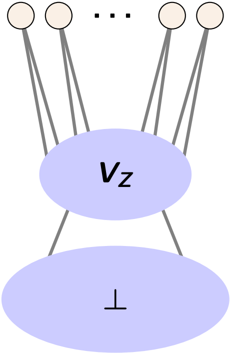


Local complementations over specific sets of vertices ?



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- An infinite strict hierarchy of local equivalence;
- LU=LC for graph states up to 19 qubits;
- A quasi-polynomial algorithm for LU-equivalence.

Some open questions

- Does there exist a counter-example to the LU=LC conjecture between 20 and 26 qubits ?

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- Does there exist a polynomial-time algorithm for LU-equivalence ?

Thanks

arXiv:2409.20183

arXiv:2502.06566