Deciding Local Unitary Equivalence of Graph States in Quasi-Polynomial Time

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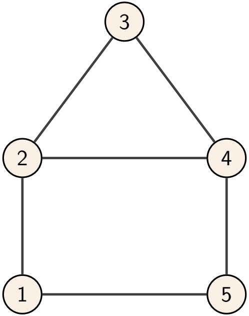
Graph states and local complementation

Graph states

A graph state is a quantum state represented by an undirected[1] and simple[2] graph. The vertices represent the qubits and the edges represent entanglement.

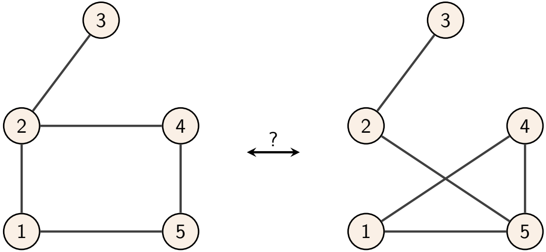


[1]Edges do not have a direction.

[2]No multiples edges and no loops.

Entanglement of graph states

Graph states are useful entangled resources (MBQC, quantum networks...) $\rightarrow$ It is a fundamental problem to know whether two graph states have the same entanglement, i.e. are **LU-equivalent** (local unitary).



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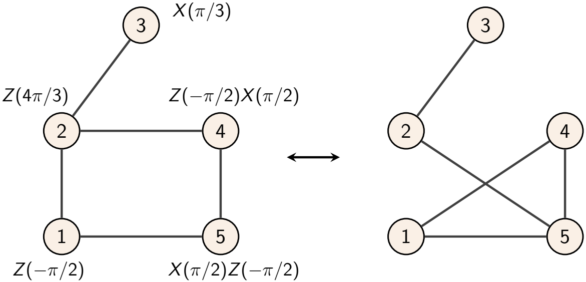
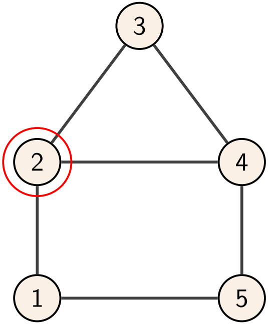


Local complementation

Definition (Kotzig, 1966)

A local complementation on a vertex u consists in complementing the (open) neighbourhood of u.



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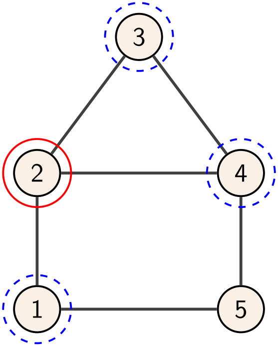

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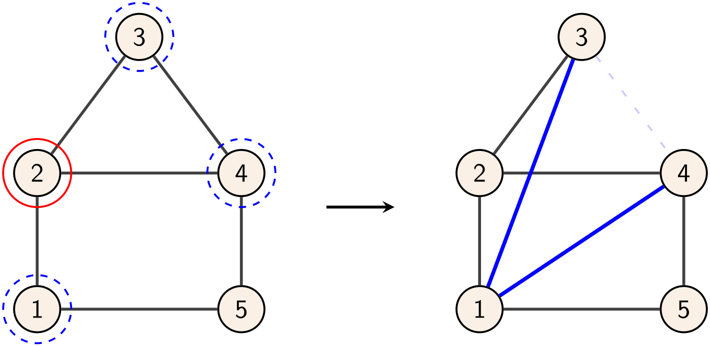
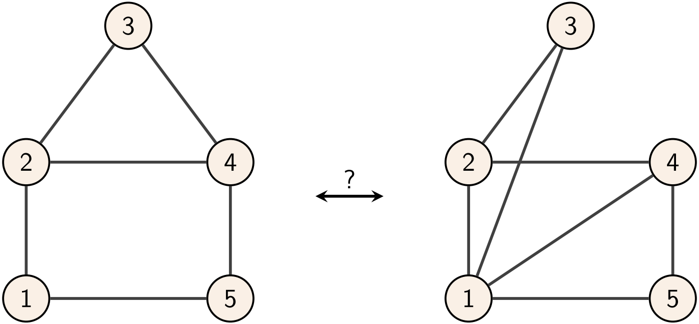


Algorithmic aspect of local complementation

Proposition (Bouchet, 1991)

There exists an efficient algorithm to recognize whether two graphs are related by a sequence local complementations.



Local complementation and entanglement

Local complementation preserves the entanglement of a graph state.

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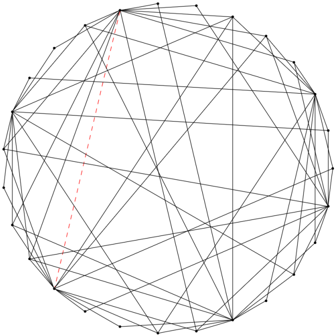
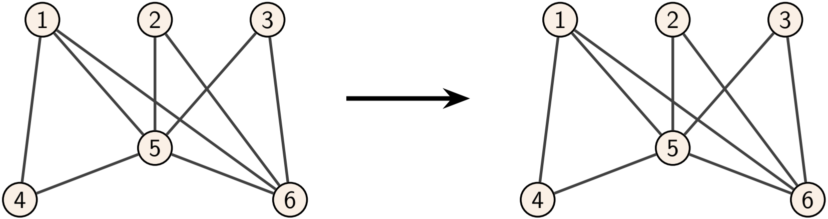
Local complementation preserves the entanglement of a graph state.
The converse is false:
$\rightarrow$ 27-qubit pair of graph states that are LU-equivalent but not related by local complementations (Ji et al. 2008).



A generalization of local complementation that captures LU-equivalence

A refinement of idempotent local complementations

A sequence of local complementations may leave the graph invariant.



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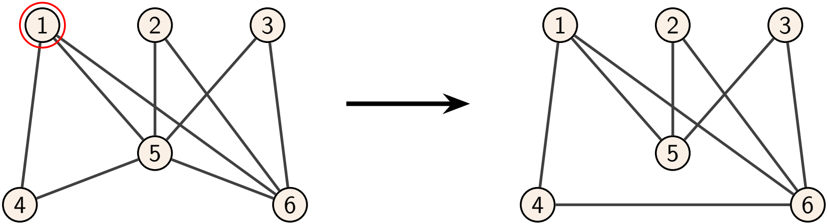

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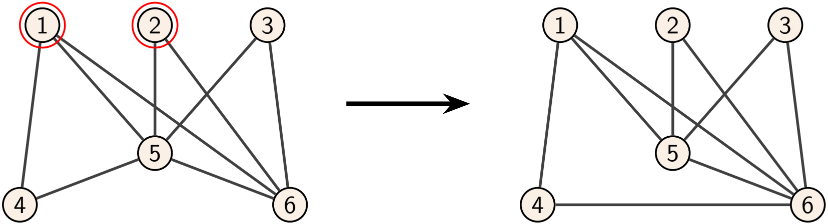

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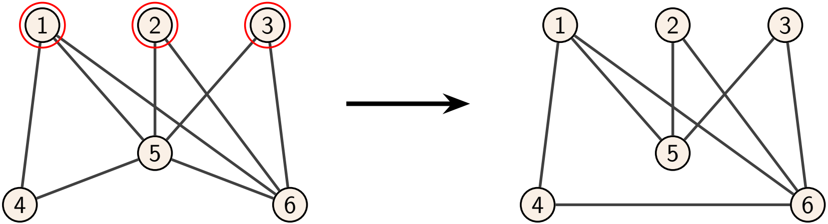

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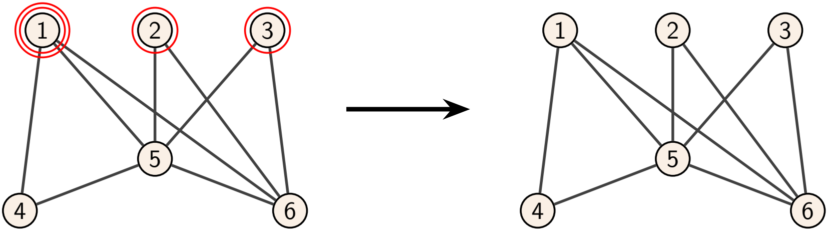

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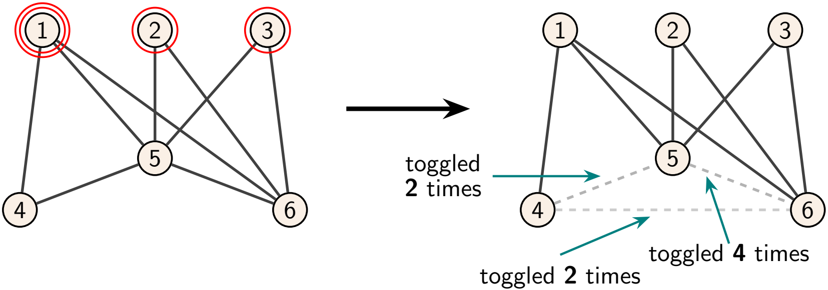

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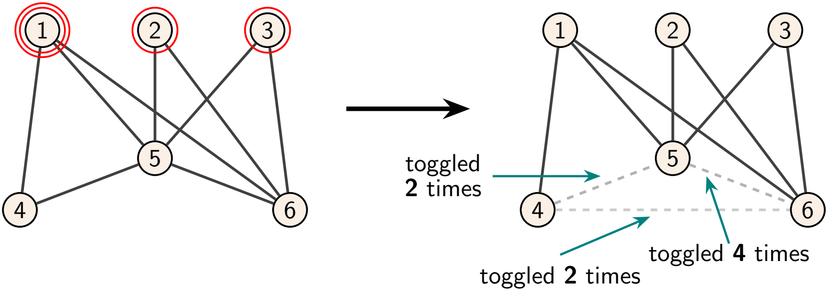

A **2-local complementation** consists in toggling every edge that was toggled 2 mod 4 times by the idempotent local complementations.

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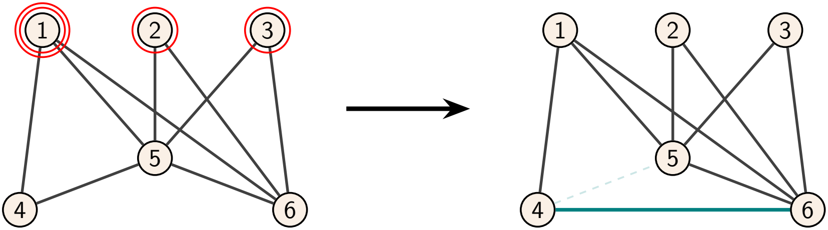

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r-local complementation

$\rightarrow$ 3-local complementation is defined as a refinement of idempotent 2-local complementation

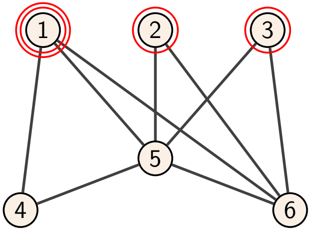
$\rightarrow$ 4-local complementation is defined as a refinement of idempotent 3-local complementation

...

$\rightarrow$ r-local complementation is defined as a refinement of idempotent $(r-1)$-local complementation

Conditions for r-local complementation

A 2-local complementation over S is valid if every pair or triplet of vertices outside of S has an even number of common neighbors in S.



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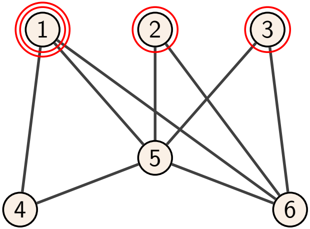


In general, there are $O(n^{r+1})$ parity conditions to check for a r-local complementation to be valid.

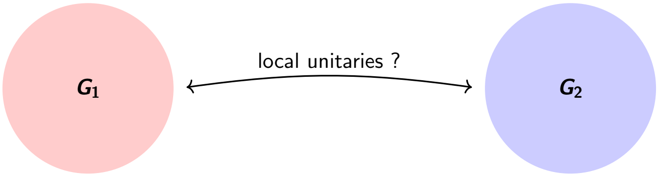
r-local complementation and LU-equivalence

Theorem (C., Perdrix 2024)

Two LU-equivalent graph states a related by a sequence of local complementations containing a single **r-local complementation**.

An algorithm for LU-equivalence

Standard form



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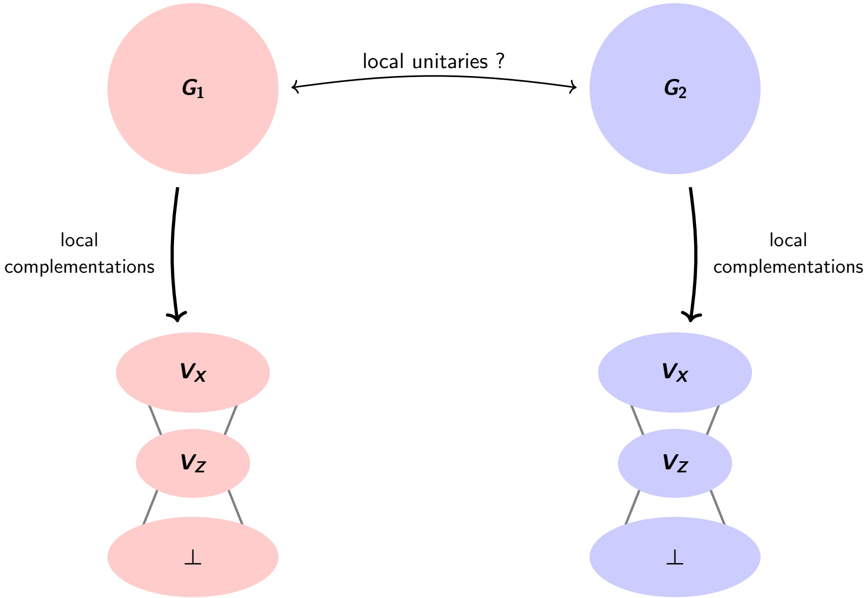

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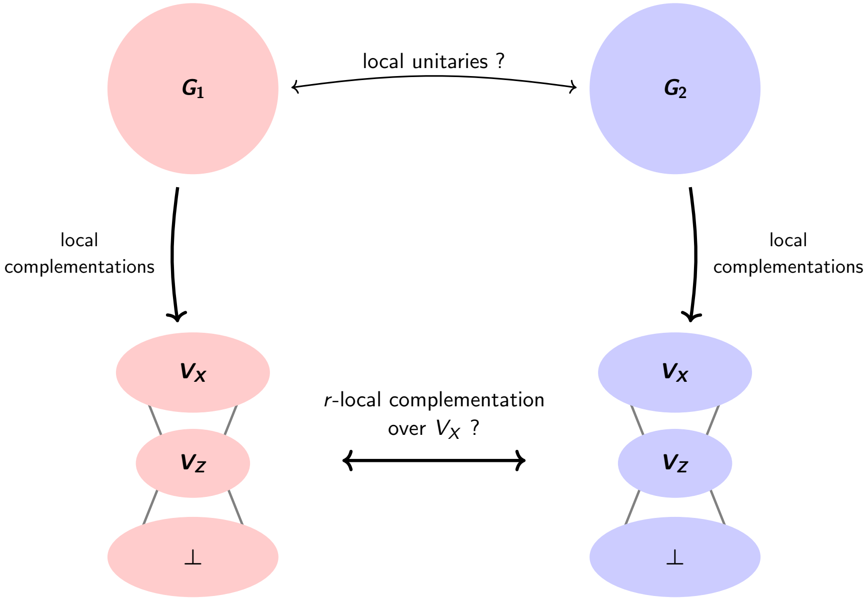
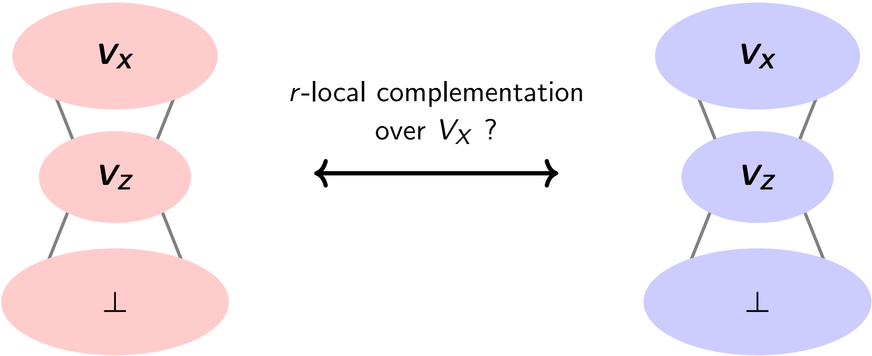


Solving for r-local complementation



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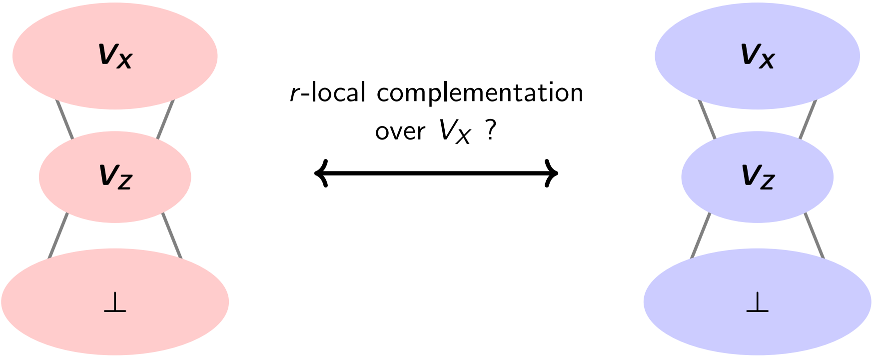

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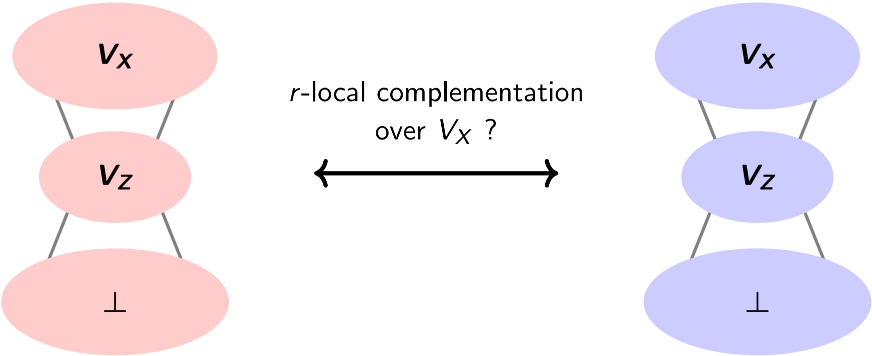

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Deciding LU-equivalence is done in complexity $O(n^{r+1})$.

Bounding r

Proposition (this work)

Two LU-equivalent n-qubit graph states a related by a sequence of local complementations containing a single r-local complementation where

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Theorem (this work)

There exists an algorithm that decides whether two graph states are LU-equivalent with runtime $n^{\log_2(n)+O(1)}$.

Graphs for which local complementation captures LU-equivalence

Bounds for LU=LC

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Corollary

LU=LC up to 15 qubits.

Improving the bound for LU=LC

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Proposition (this work)

LU=LC for graph states up to 19 qubits.

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- Can we make it polynomial ?
- LU=LC up to 26 qubits ?

Thanks