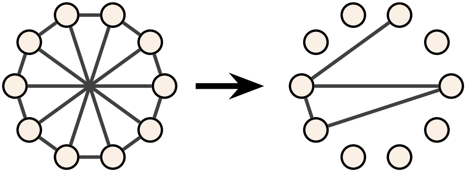
Vertex-minor universal graphs for generating entangled quantum subsystems

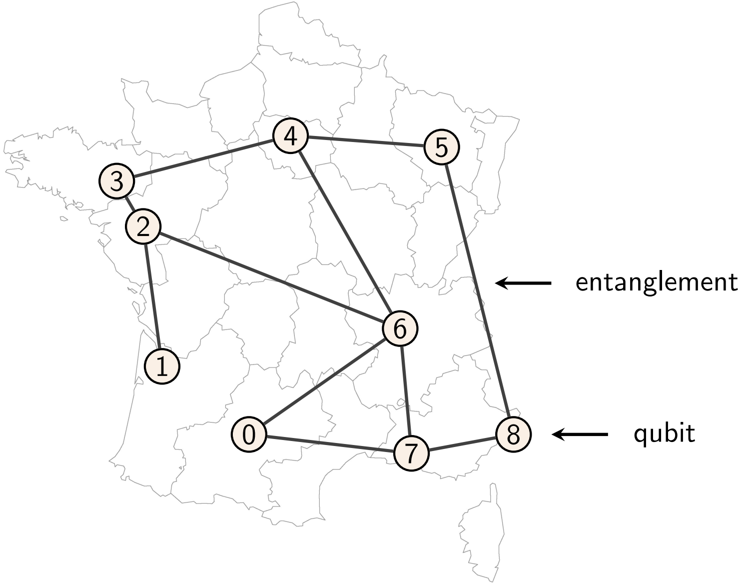
Maxime Cautrès, Nathan Claudet, Mehdi Mhalla, Simon Perdrix, Valentin Savin, Stéphan Thomassé

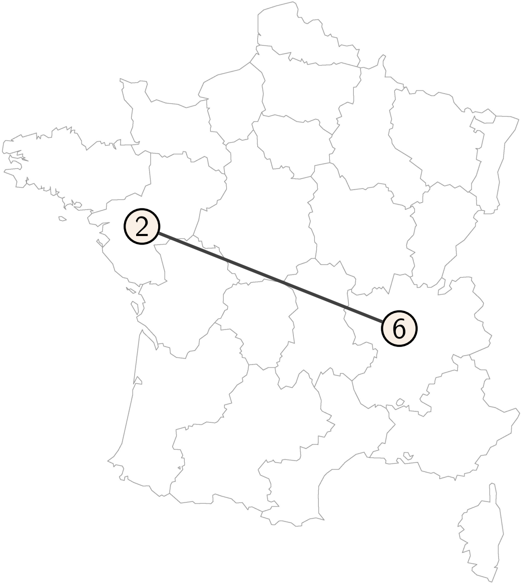
ICALP2024 - 12/07/24
arXiv:2402.06260



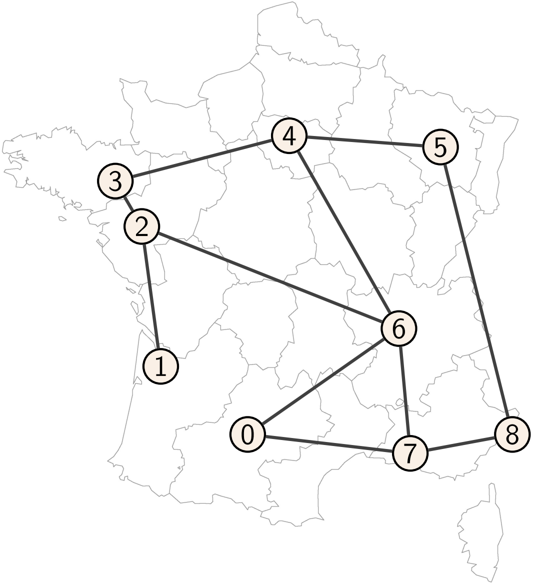
Motivation - quantum communication networks

Quantum communication networks

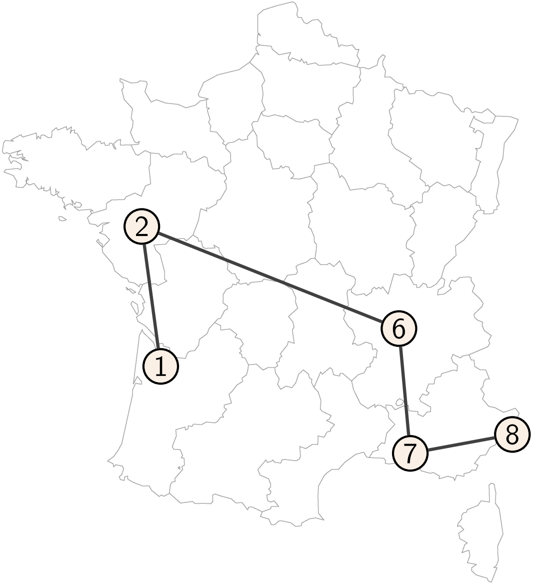



2
6

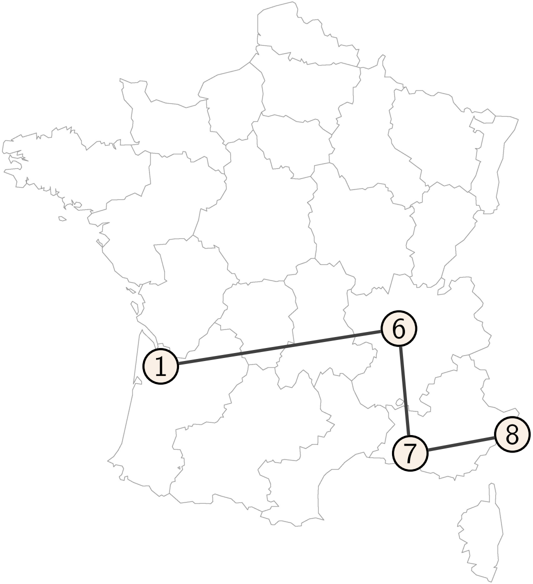
Quantum communication networks



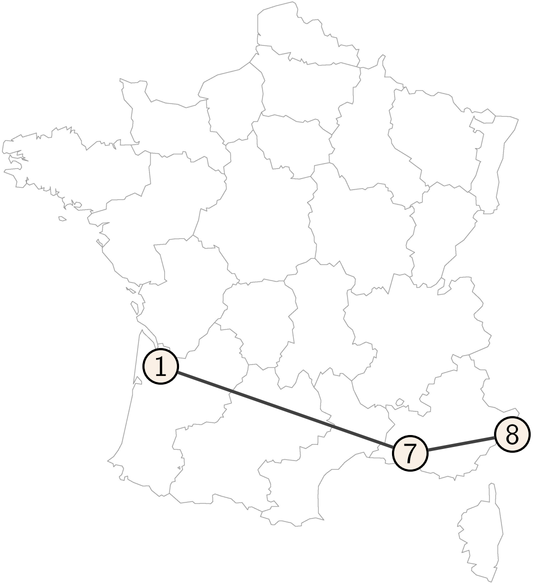
Quantum communication networks



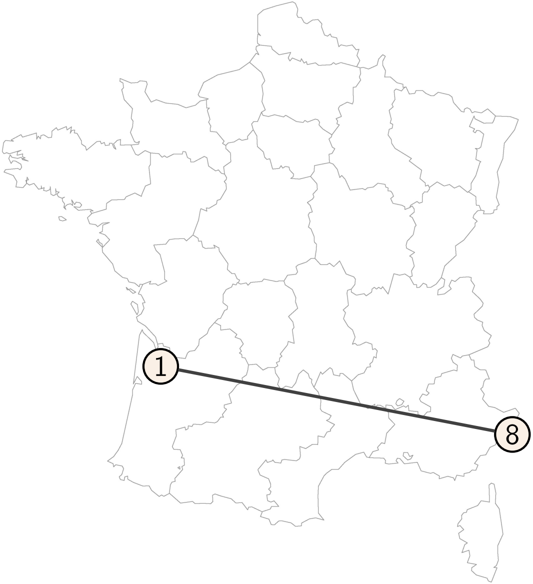
Quantum communication networks



Quantum communication networks



Quantum communication networks



Problem: Generating arbitrary graph states

Can we create an EPR-pair between any two nodes using only local operations (and classical communication)?

Problem: Generating arbitrary graph states

Can we create an EPR-pair between any two nodes using only local operations (and classical communication)?

Yes, if (and only if) the graph is connected.

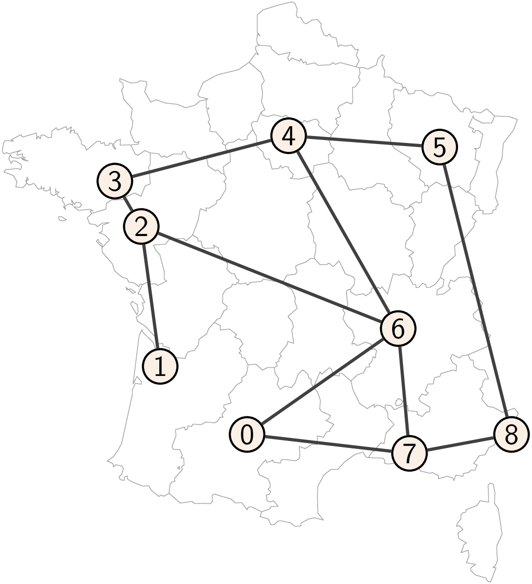
Problem: Generating arbitrary graph states

Can we create an EPR-pair between any two nodes using only local operations (and classical communication)?

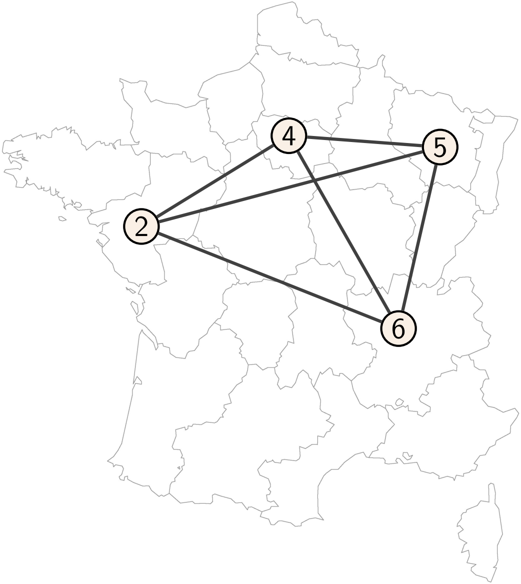
Yes, if (and only if) the graph is connected.

Natural question: What if we want to create any arbitrary **graph state** between any nodes?

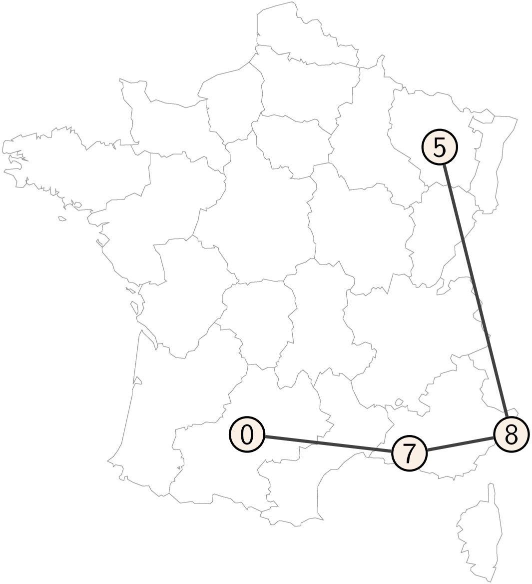
Creating graph states



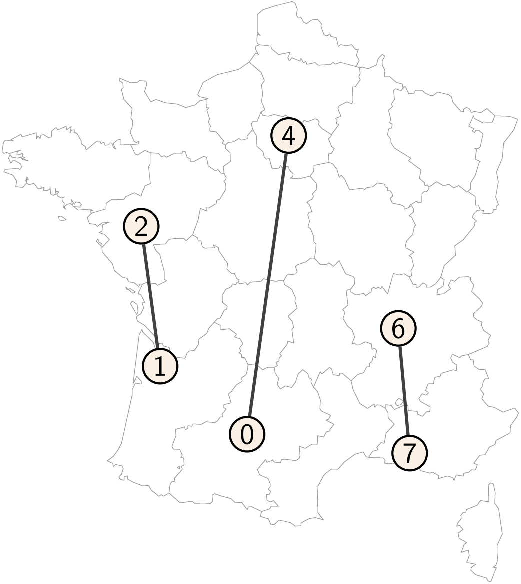
Creating graph states



Creating graph states



Creating graph states

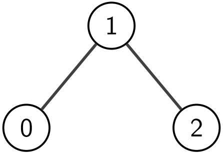


Graph states

Definition (Graph state)

Given an undirected and simple graph $G = (V, E)$, a graph state $|G\rangle$ is a quantum state written as :

$$|G\rangle = \left(\prod_{(u,v)\in E} CZ_{u,v} \right) |+\rangle_V$$

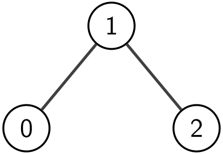


Graph states

Definition (Graph state)

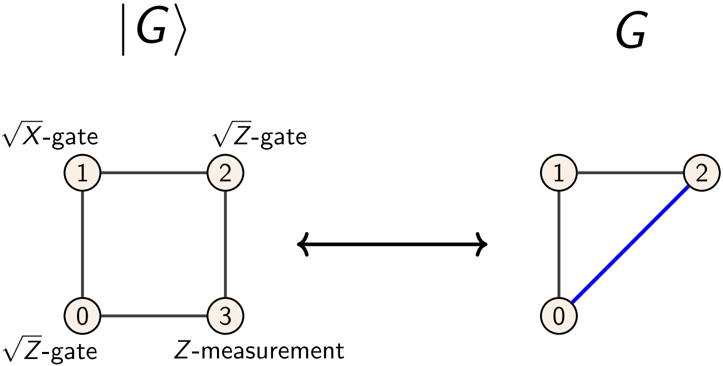
Given an undirected and simple graph $G = (V, E)$, a graph state $|G\rangle$ is a quantum state written as :

$$|G\rangle = \left(\prod_{(u,v)\in E} CZ_{u,v} \right) |+\rangle_V$$



$$\begin{aligned}|G\rangle &= CZ_{0,1} CZ_{1,2} \left(|+\rangle_0 \otimes |+\rangle_1 \otimes |+\rangle_2 \right) \\ &= \frac{1}{\sqrt{8}} \left(|000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle \right)\end{aligned}$$

Correspondence between graph states and graphs



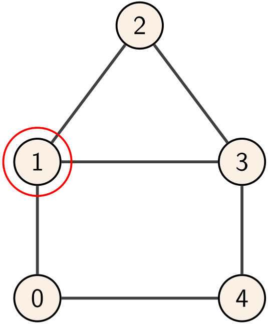
local (i.e. single-qubit)
quantum operations

vertex deletions
& local complementations

Local complementation

Definition

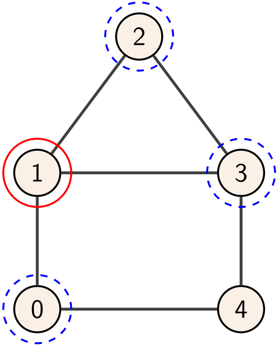
A local complementation on a vertex u consists in complementing the (open) neighborhood of u.



Local complementation

Definition

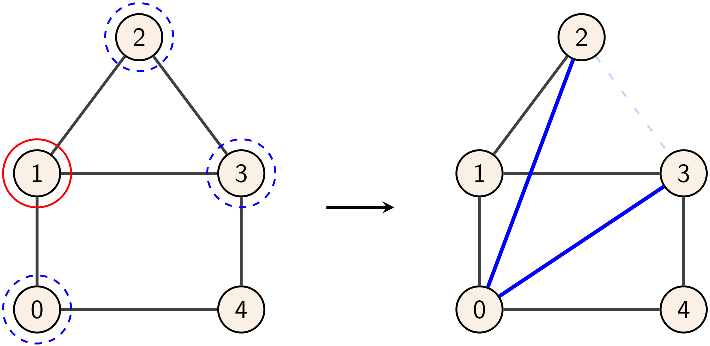
A local complementation on a vertex u consists in complementing the (open) neighborhood of u.



Local complementation

Definition

A local complementation on a vertex u consists in complementing the (open) neighborhood of u.

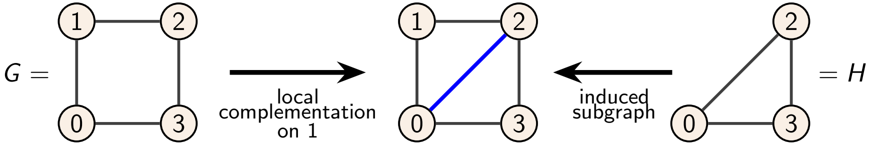


Vertex-minor universal graphs: definition

Vertex-minors

Definition (Vertex-minor)

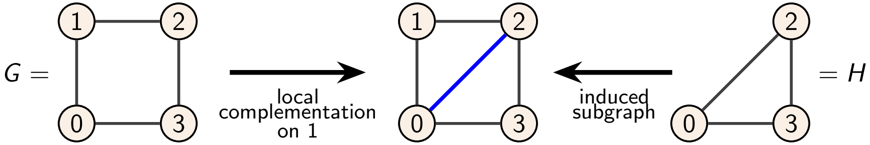
Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ such that $V_H \subseteq V_G$, H is a vertex-minor of G if H can be obtained as a induced subgraph of G by means of local complementations.



Vertex-minors

Definition (Vertex-minor)

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ such that $V_H \subseteq V_G$, H is a vertex-minor of G if H can be obtained as a induced subgraph of G by means of local complementations.

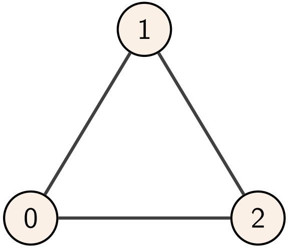


Definition

A graph G is *k-vertex-minor universal* if any graph on any k vertices is a vertex-minor of G.

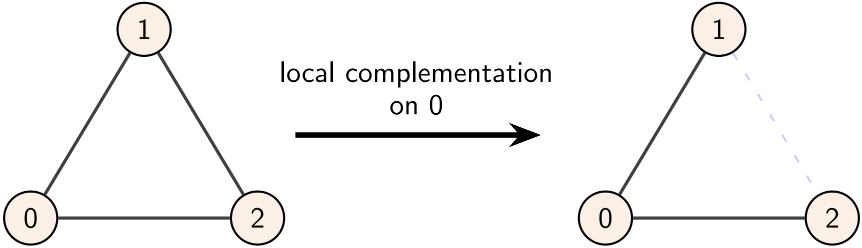
k-vertex-minor universal graphs : example 1

K_3 is 2-vertex-minor universal.



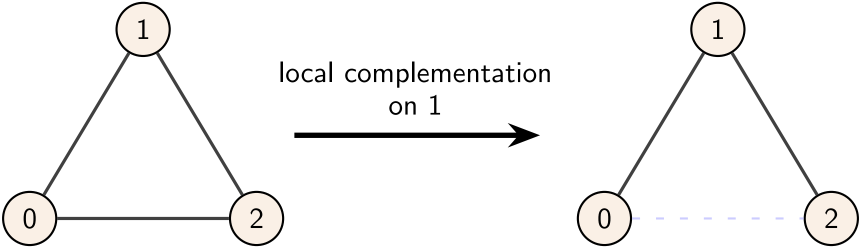
k-vertex-minor universal graphs : example 1

K_3 is 2-vertex-minor universal.



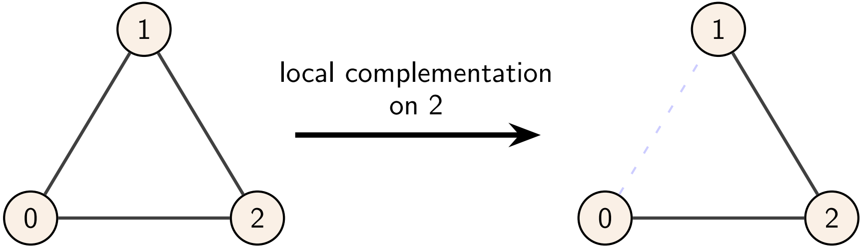
k-vertex-minor universal graphs : example 1

K_3 is 2-vertex-minor universal.



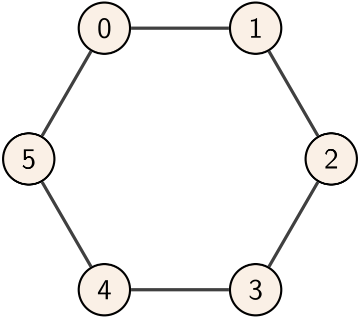
k-vertex-minor universal graphs : example 1

K_3 is 2-vertex-minor universal.



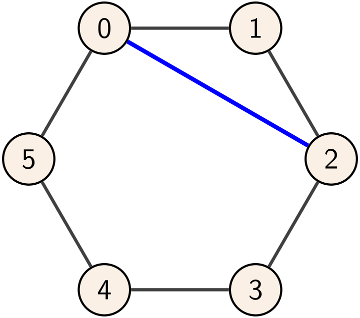
k-vertex-minor universal graphs : example 2

C_6 is 3-vertex-minor universal.



k-vertex-minor universal graphs : example 2

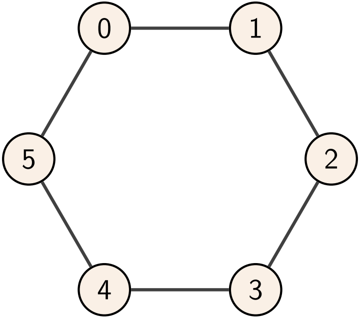
C_6 is 3-vertex-minor universal.



To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.

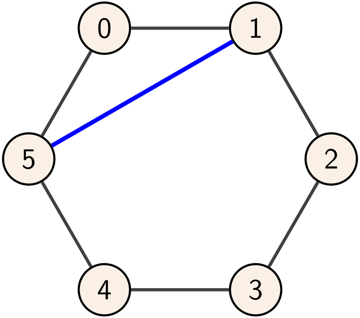
k-vertex-minor universal graphs : example 2

C_6 is 3-vertex-minor universal.



To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.

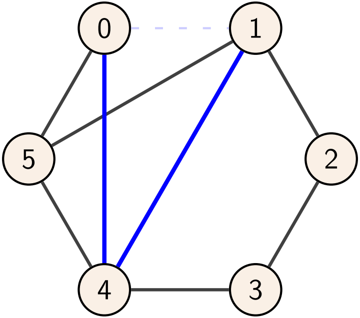
C_6 is 3-vertex-minor universal.



To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
To induce the empty graph on $\{0, 1, 2\}$: Local complementation on 0,

k-vertex-minor universal graphs : example 2

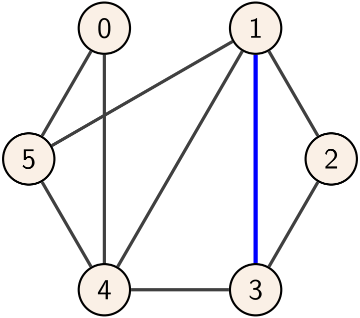
C_6 is 3-vertex-minor universal.



To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
To induce the empty graph on $\{0, 1, 2\}$: Local complementation on 0, on 5,

k-vertex-minor universal graphs : example 2

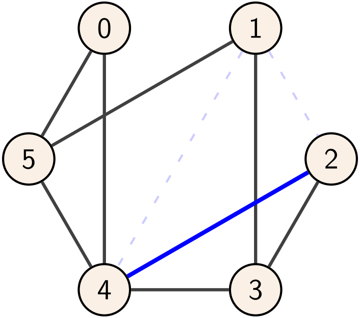
C_6 is 3-vertex-minor universal.



To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
To induce the empty graph on $\{0, 1, 2\}$: Local complementation on 0, on 5, on 2,

k-vertex-minor universal graphs : example 2

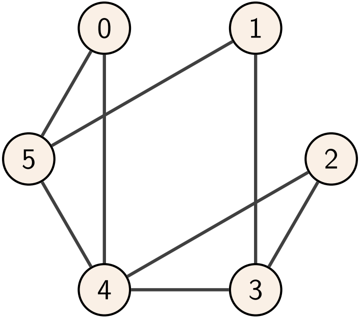
C_6 is 3-vertex-minor universal.



To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
To induce the empty graph on $\{0, 1, 2\}$: Local complementation on 0, on 5, on 2, on 3.

k-vertex-minor universal graphs : example 2

C_6 is 3-vertex-minor universal.



To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
To induce the empty graph on $\{0, 1, 2\}$: Local complementation on 0, on 5, on 2, on 3.

Quantum counterpart

Proposition

If G is k-vertex-minor universal, any graph state on any k qubits of $|G\rangle$ can be induced by local operations and classical communication.

An upper bound on k-vertex-minor universality

For an arbitrary k, existence of k-vertex-minor universal graphs ?

An upper bound on k-vertex-minor universality

For an arbitrary k, existence of k-vertex-minor universal graphs ? Of reasonable size ?

An upper bound on k-vertex-minor universality

For an arbitrary k, existence of k-vertex-minor universal graphs ? Of reasonable size ?

A lower bound:

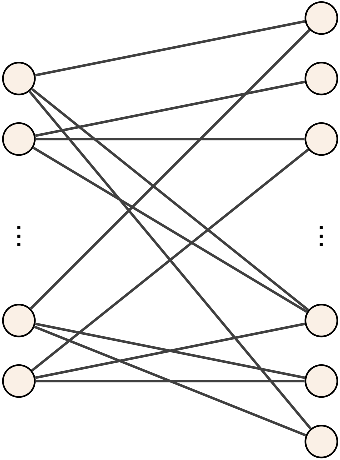
Proposition

A k-vertex-minor universal graph is of order $\Omega(k^2)$.

Existence of (random) k-vertex-minor universal graphs of order $\Theta(k^2)$

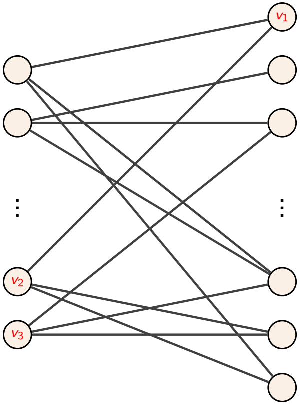
Outline of the construction

Random bipartite graph $G = (L \cup R, E)$ (the probability of an edge existing between L and R is $1/2$). $|L| = \Theta(k \ln(k))$, $|R| = \Theta(k^2)$.



Proof of vertex-minor universality: greedy algorithm

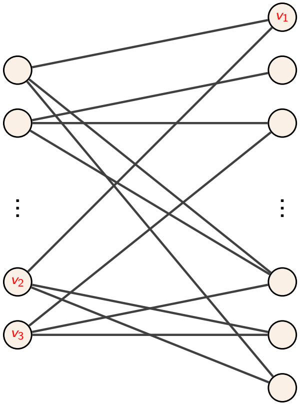
Given any fixed set of k vertices:



Proof of vertex-minor universality: greedy algorithm

Given any fixed set of k vertices:

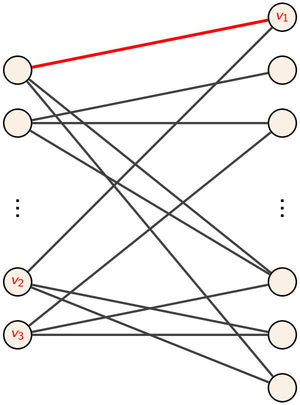
- 1 - Move every vertex to the left by means of pivoting.



Proof of vertex-minor universality: greedy algorithm

Given any fixed set of k vertices:

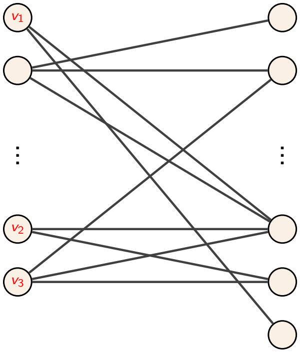
- 1 - Move every vertex to the left by means of pivoting.



Proof of vertex-minor universality: greedy algorithm

Given any fixed set of k vertices:

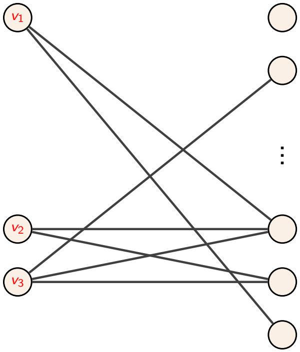
- 1 - Move every vertex to the left by means of pivoting.



Proof of vertex-minor universality: greedy algorithm

Given any fixed set of k vertices:

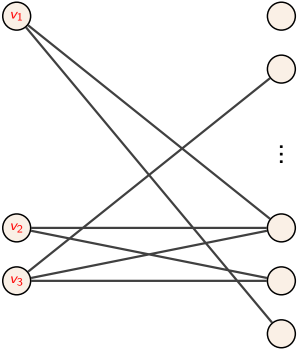
- 1 - Move every vertex to the left by means of pivoting.



Proof of vertex-minor universality: greedy algorithm

Given any fixed set of k vertices:

- 1 - Move every vertex to the left by means of pivoting.
- 2 - Check if the incidence matrix if of full rank.

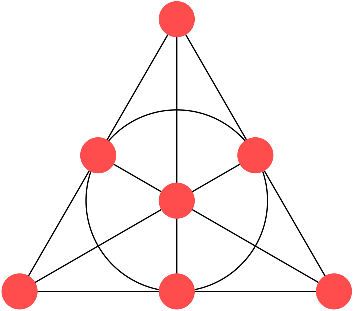


$$
\begin{array}{ccc}
(v_1, v_2) & (v_1, v_3) & (v_2, v_3)
\end{array}
$$

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

Explicit construction of k-vertex-minor universal graphs of order $O(k^4)$

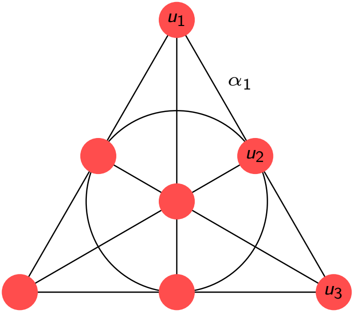
Projective planes



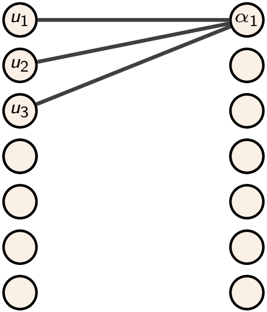
PG$(2, q)$: projective plane over $\mathbb{F}_q$

Key feature: Any two distinct lines intersect in one unique point, and for any two distinct points there is one unique line containing them.

Projective planes



$\mathrm{PG}(2, q)$: projective plane over $\mathbb{F}_q$



G_q: bipartite incidence graph
graph of $\mathrm{PG}(2, q)$

Key feature: Any two distinct lines intersect in one unique point, and for any two distinct points there is one unique line containing them.

Vertex-minor universal graphs from projective planes

Theorem

G_q (of order $\Theta(q^2)$) is $\Omega(\sqrt{q})$-vertex-minor universal.

Conclusion

Conclusion

k-vertex-minor universal graphs:

- Probabilistic construction of order $\Theta(k^2)$.
- Explicit construction of order $O(k^4)$.

Future work:

Conclusion

k-vertex-minor universal graphs:

- Probabilistic construction of order $\Theta(k^2)$.
- Explicit construction of order $O(k^4)$.

Future work:

- Better explicit construction of k-vertex-minor universal graphs.

Conclusion

k-vertex-minor universal graphs:

- Probabilistic construction of order $\Theta(k^2)$.
- Explicit construction of order $O(k^4)$.

Future work:

- Better explicit construction of k-vertex-minor universal graphs.
- Quantum-wise: what if we allow more than 1 qubit per party ?

Thanks

arXiv:2402.06260