

Quantum graph states for graph theorists

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21/04/26

Georgia Tech - Graph theory seminar



Quantum computing

Computing with quantum physics

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Theorem (Shor's algorithm, 1994)

*There exist an **efficient** quantum algorithm that finds the prime factors of an integer.*

$$15 = 5 \times 3$$

$$221 = 13 \times 17$$

$$269535011 = 12923 \times 20857$$

How to build a quantum computer

- In classical computing, bits: 0 or 1.
- In quantum computing, quantum bits (qubits): $|0\rangle$, $|1\rangle$, but also $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

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+ ways to interact with the qubits to create **entanglement**.

Quantum entanglement

- Quantum states are composed of qubits.



$$|0\rangle \otimes |0\rangle = |00\rangle$$

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Quantum entanglement

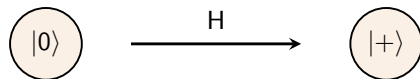
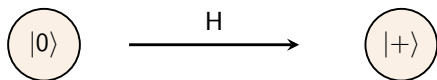
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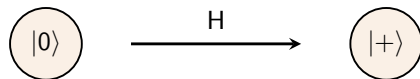
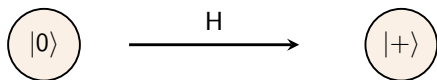


$$|0\rangle \otimes |0\rangle = |00\rangle$$

$$\begin{aligned} & |+\rangle \otimes |+\rangle \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \end{aligned}$$

Quantum entanglement

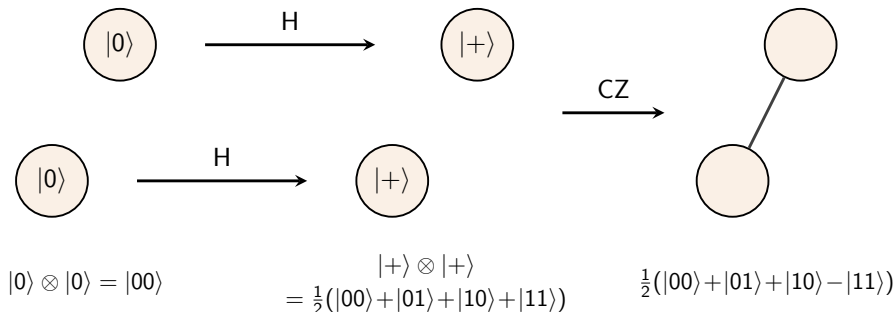
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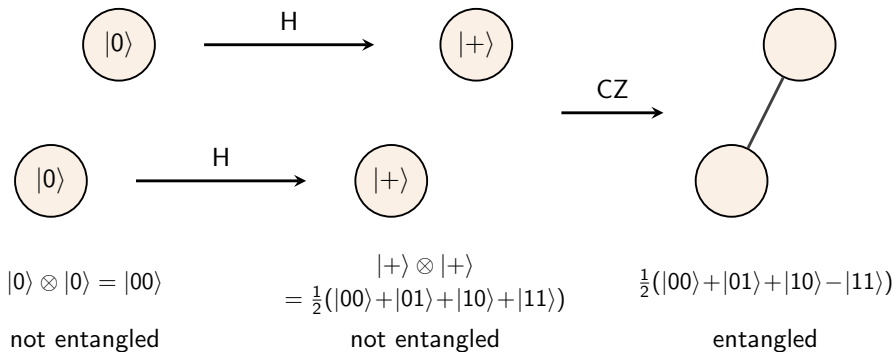
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Qubits are **entangled** when they cannot be described as separate entities.

Possible applications of quantum computing

- Optimisation

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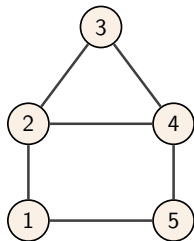
- Optimisation
- Simulation of quantum systems (drug discovery, material science...)
- Communication & cryptography (safe encryption methods, key distribution...)

Graph states: definition

Graphs

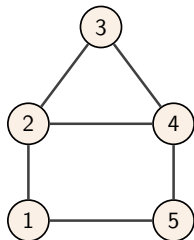
Definition

A graph is composed of two sets, a set $V \in \mathbb{N}$ of (labeled) vertices and a set $E \in V^2$ of edges linking vertices.



Definition

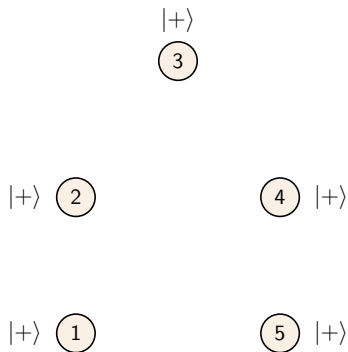
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- We consider graphs that are simple (no multiples edges, no loops) and undirected (edges do not have a direction).

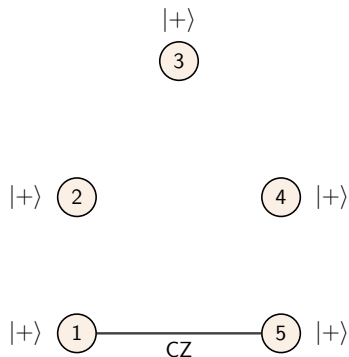
Graph states

A graph state is a quantum state represented by a graph. The vertices represent the qubits and the edges represent entanglement.



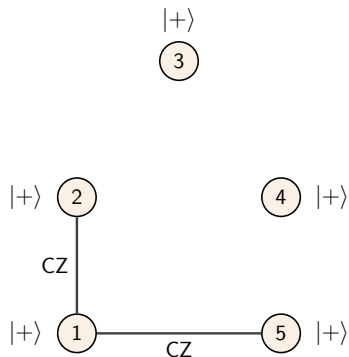
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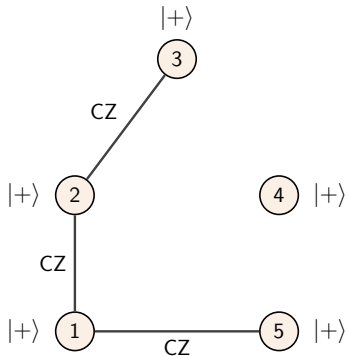
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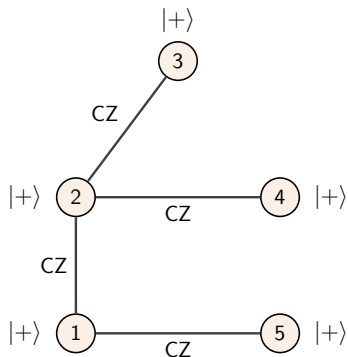
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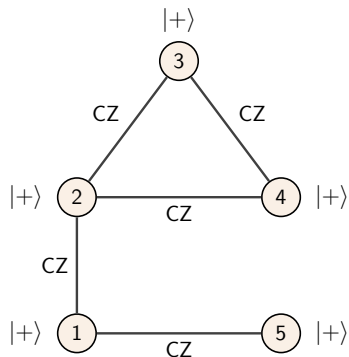
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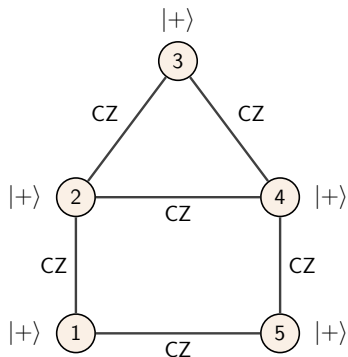
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Proposition

Graph states are in one-to-one correspondance with (undirected, simple) graphs.

Structure of the presentation

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- Application 1: Quantum networks.
- Application 2: Measurement-based quantum computing (MBQC).
- Classifying graph states.

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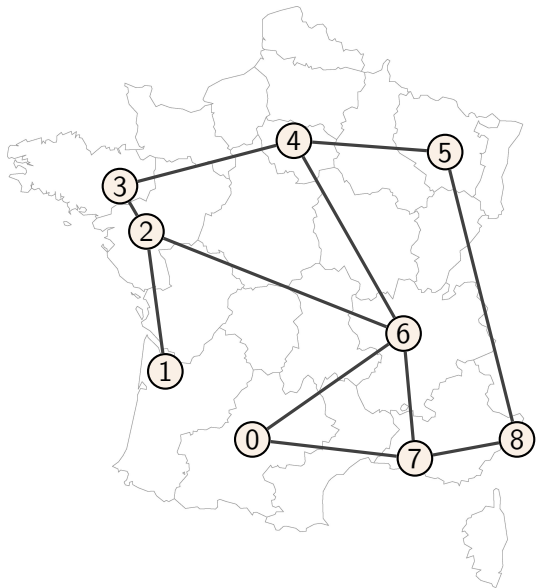
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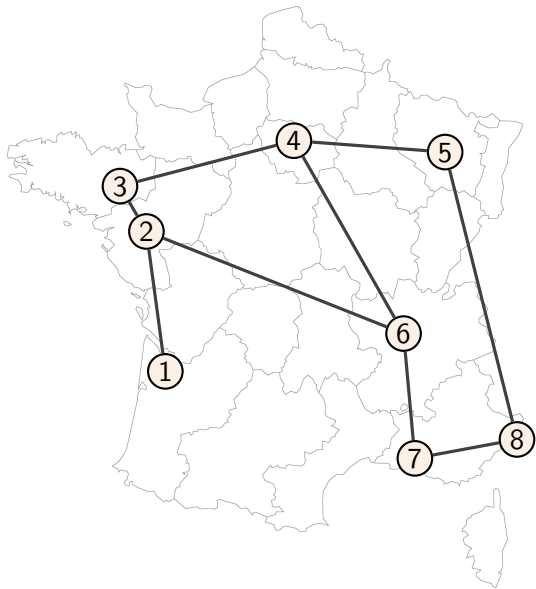
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When are two graph states equivalent?

Quantum networks

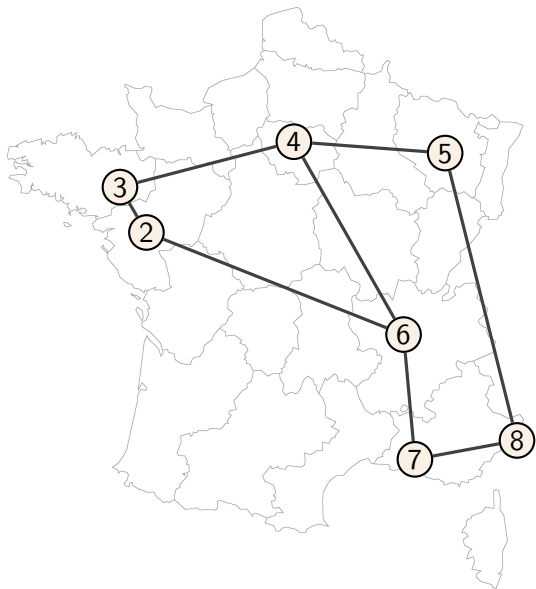
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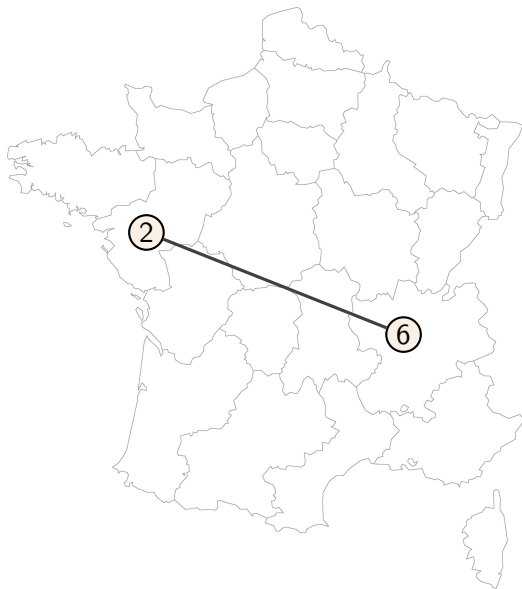
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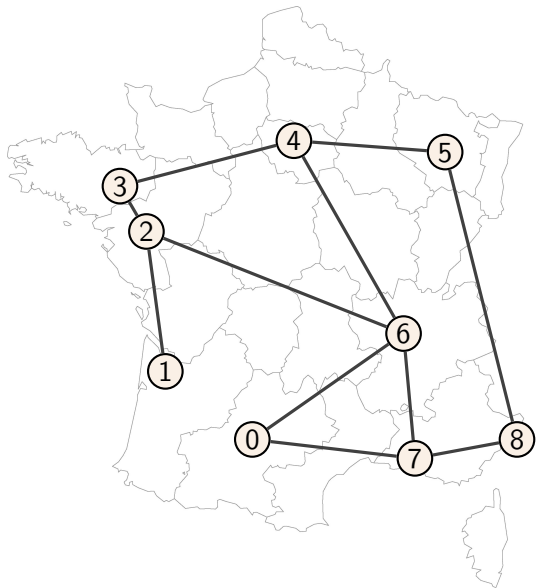
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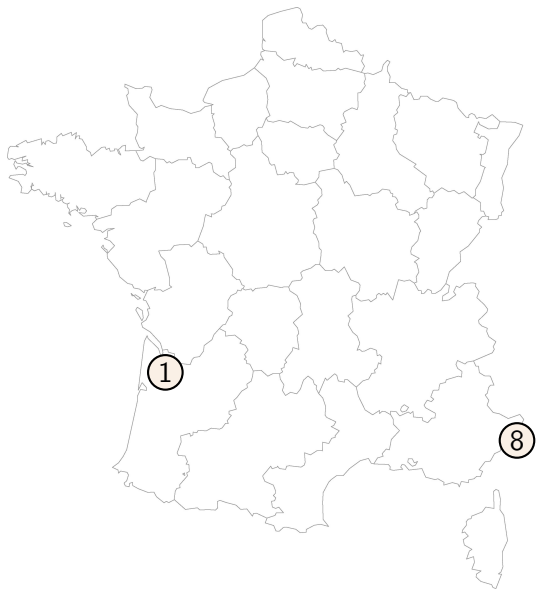
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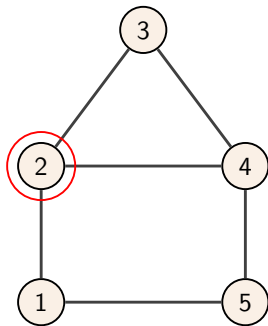


Local complementation

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Definition (Kotzig, 1966)

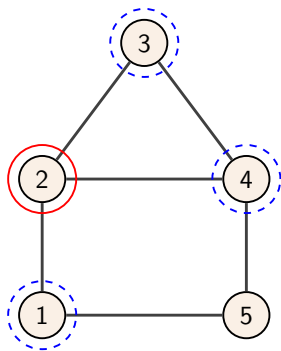
A local complementation on a vertex u consists in complementing the (open) neighbourhood of u .



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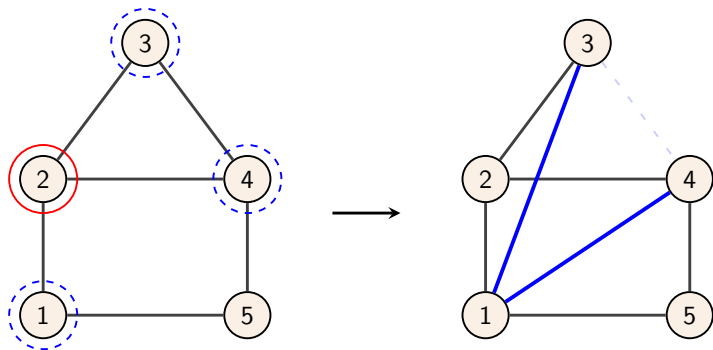
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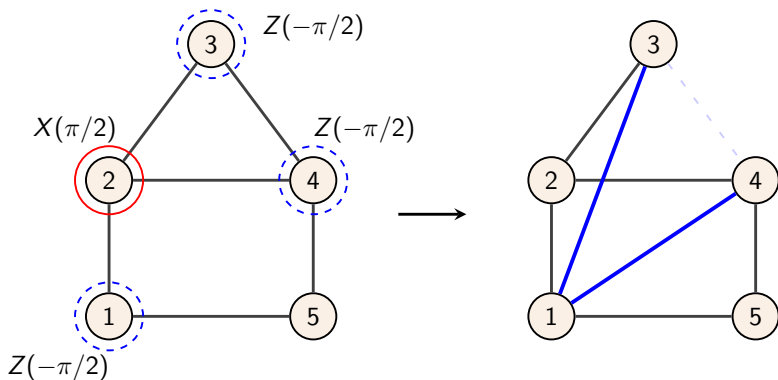
Theorem (Bouchet, 1991)

There exists an efficient algorithm to decide if two graphs are related by local complementations.

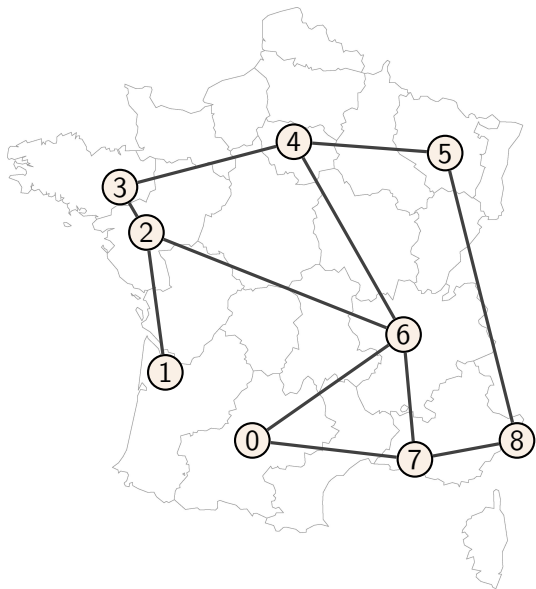
Local complementation on a graph state

Proposition (Van den Nest et al., 2004)

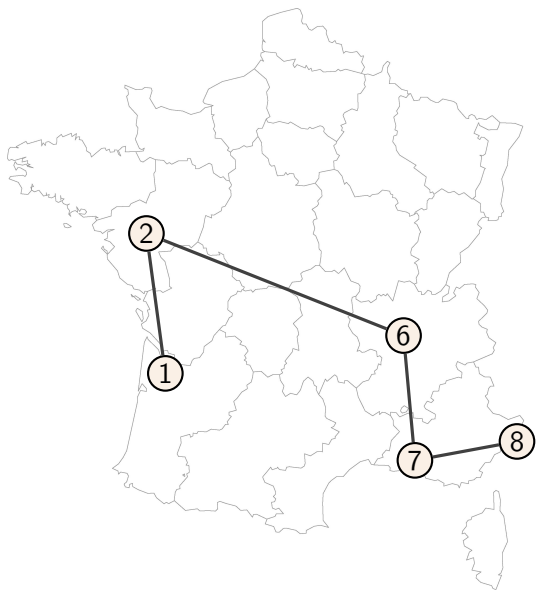
Local complementation can be implemented on a graph with local quantum operations.



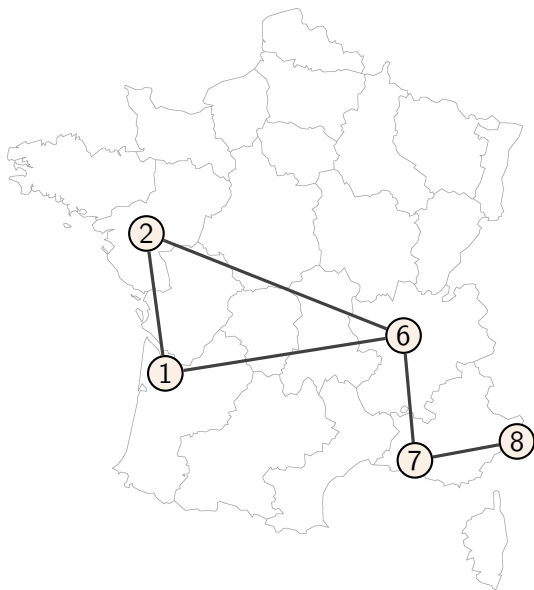
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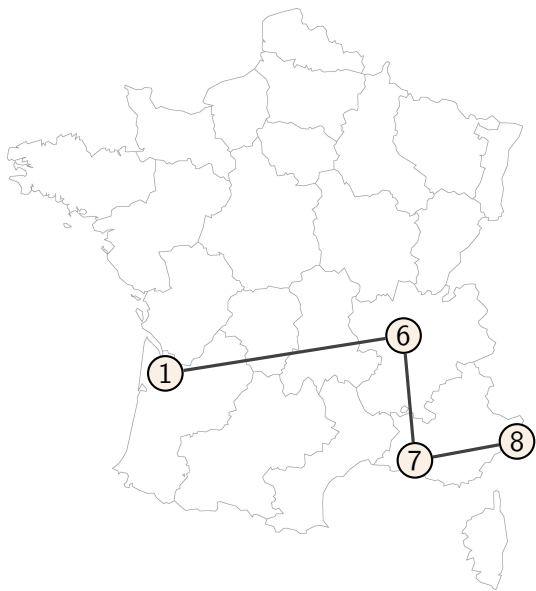
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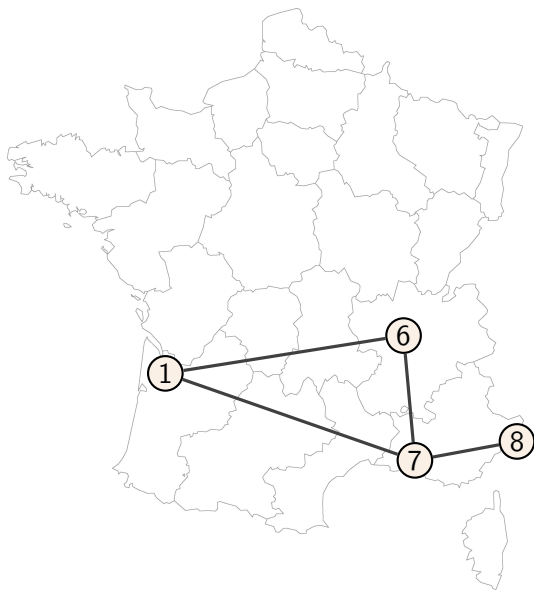
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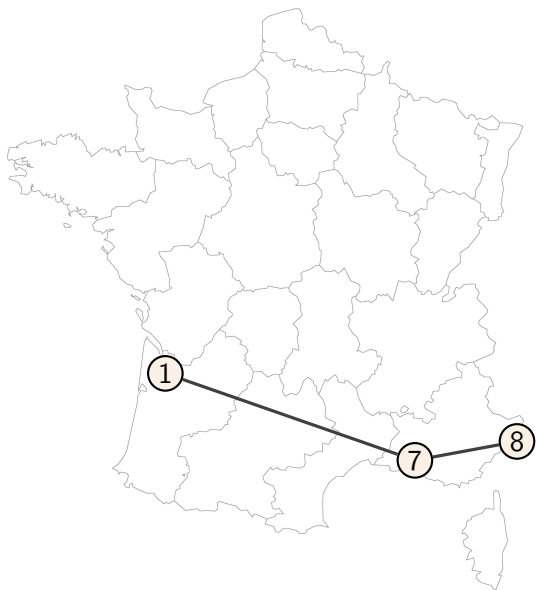
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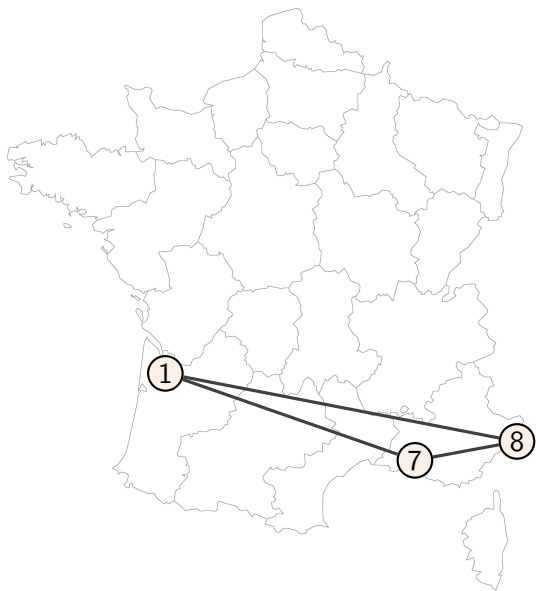
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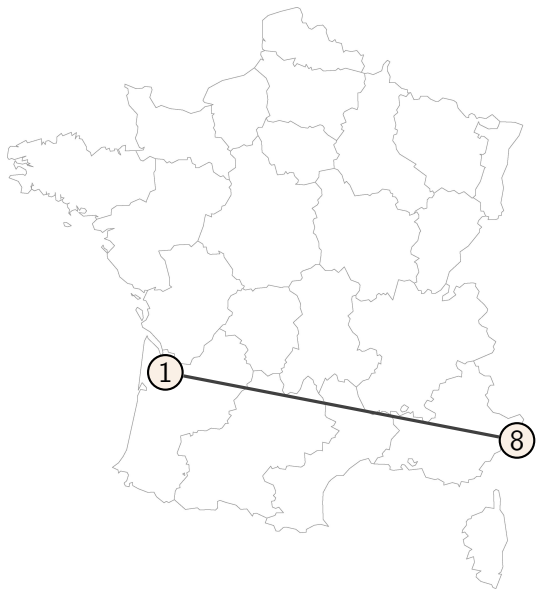
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Application: quantum communication networks



EPR-pair in a connected graph state

Proposition

If the graph state corresponds to a connected graph, an EPR-pair (i.e. the two-qubit connected graph state) between any two nodes of the network can be created using only local quantum operations.

Vertex-minors

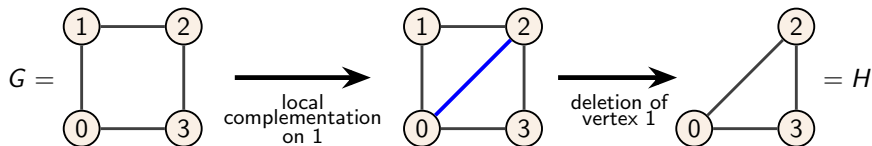
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Definition (Vertex-minor)

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ such that $V_H \subseteq V_G$, H is a vertex-minor of G if H can be obtained from G by local complementations and vertex deletions.

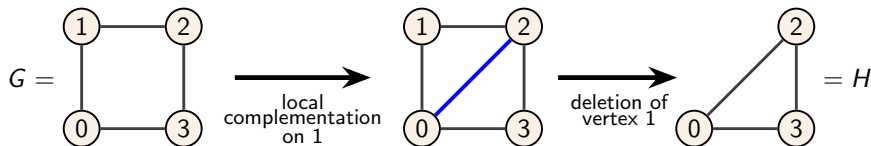


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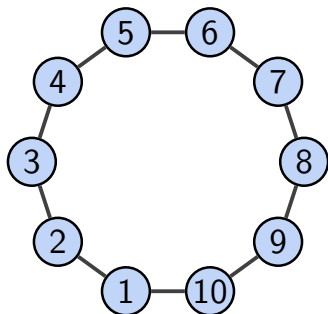
Theorem (Dahlberg, Wehner, 2018)

The graph state $|H\rangle$ can be obtained from the graph state $|G\rangle$ by local quantum operations if and only if H is a vertex-minor of G .*

Insight on quantum networks via vertex-minors

Theorem (Hahn, Dahlberg, Eisert, Pappa, 2022)

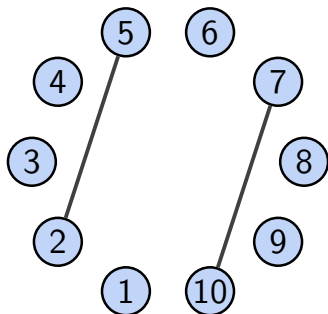
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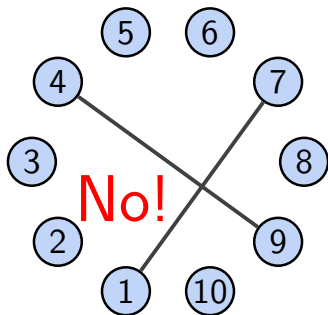
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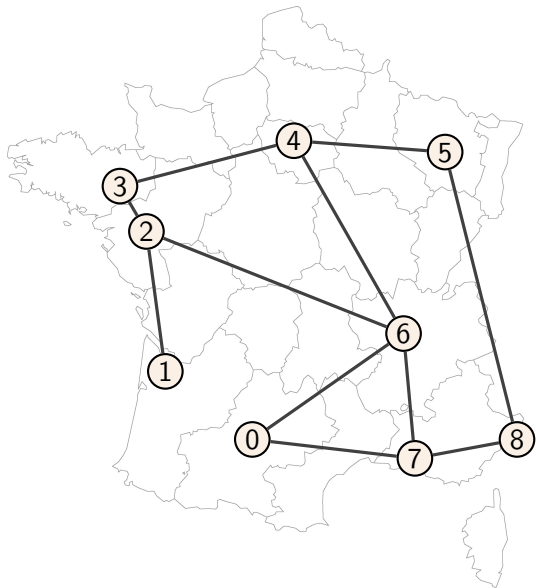
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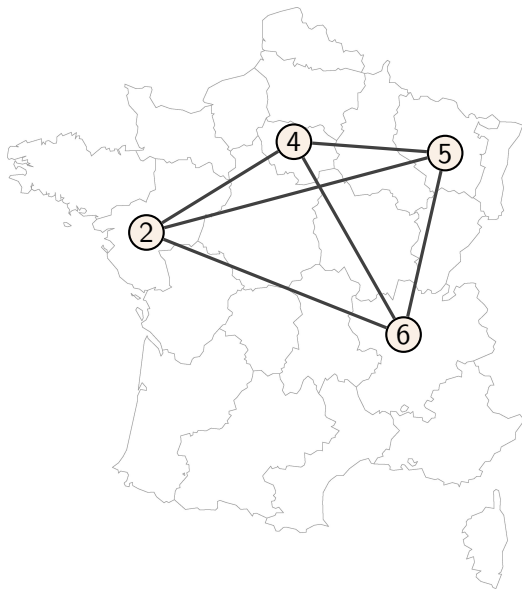
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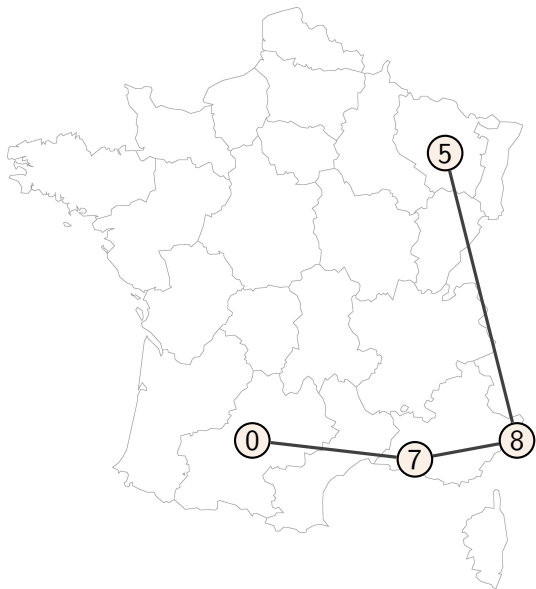
Creating arbitrary graph states



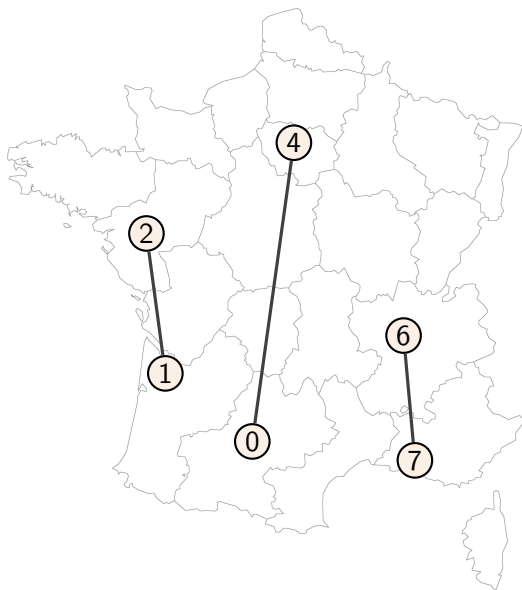
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Proposition

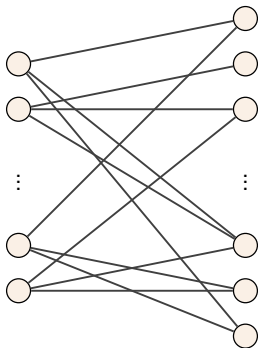
Any graph state with k qubits can be created from $|G\rangle$ with local quantum operations if and only if any graph with k vertices can be obtained as a vertex-minor of G .*

→ " k -vertex-minor universality"

Vertex-minor universality

Theorem (Cautrès, C, Mhalla, Perdrix, Savin, Thomassé, 2024)

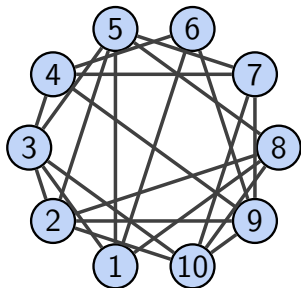
For any integer n , there exist graphs of order n that are \sqrt{n} -vertex-minor universal. This is optimal up to a multiplicative factor.



Vertex-minor universality of random graph states

Theorem (Ascoli, Frederickson, Frederickson, McFarland, Post, 2026)

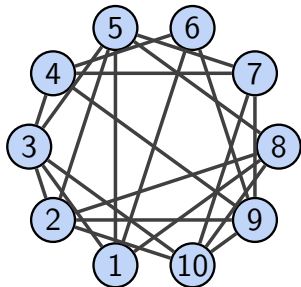
Almost all graphs are \sqrt{n} -vertex-minor universal. ($G(n, 1/2)$)



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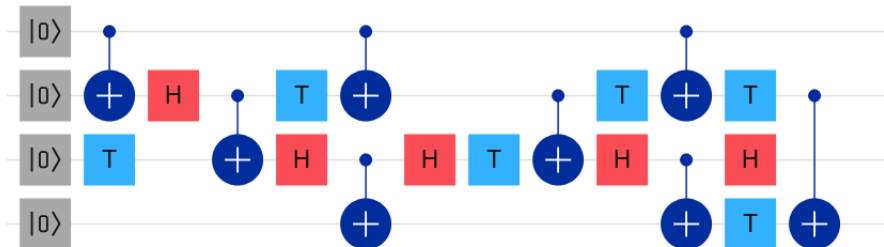


Theorem (Chao, Xu, 2026)

The graph $G(n, p)$ is \sqrt{n} -vertex-minor universal with high probability when $\Omega(\log n / \sqrt{n}) \leq p \leq 1/2$.

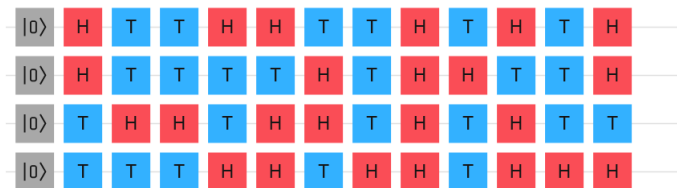
Measurement-based quantum computing

The quantum circuit model



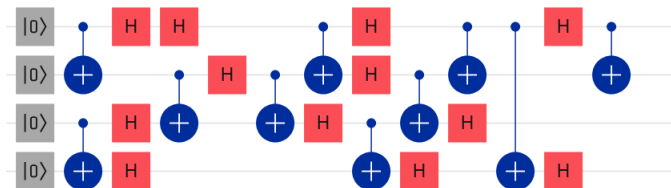
What features make quantum computers (allegedly) more powerful than classical computers?

Quantum circuits with only local gates



No entanglement \rightarrow The quantum computation can be efficiently simulated on a classical computer.

Quantum circuits with only Clifford gates



Theorem (Gottesman-Knill, 1998)

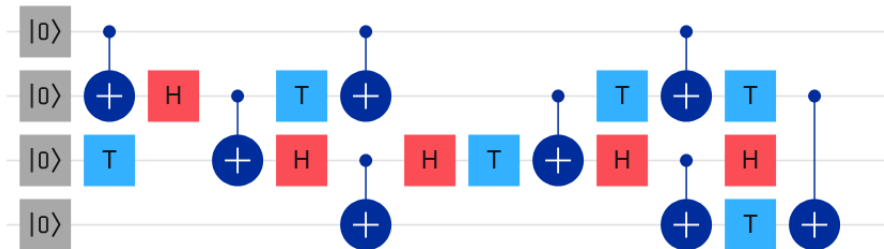
A quantum computation consisting only of Clifford gates can be efficiently simulated on a classical computer.

Sources of computational power

Two potential sources of quantum computational power:

- 1) Entanglement;
- 2) non-Clifford operations.

The quantum circuit model



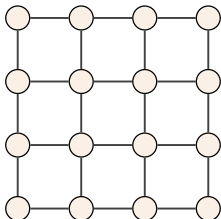
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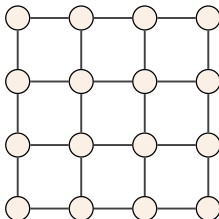
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Measurement-based quantum computing

The idea behind **measurement-based quantum computing** is to separate the two potential sources of quantum computational power.

- 1) Prepare a **graph state** with only CZ gates (Clifford gates);
- 2) **Measure** qubits in some ways that may depend on the previous measurements (no entanglement added).



"What features make quantum computers (allegedly) more powerful than classical computers?"

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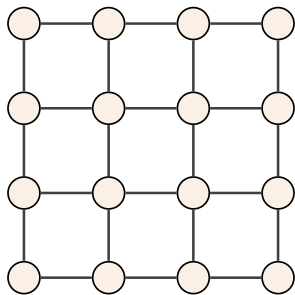
becomes

- "Which graph states allow for the same computational power as quantum circuits?"
- "Conversely, for which graph states is measurement-based quantum computing efficiently simulable on a classical computer?"

Power of measurement-based quantum computing

Theorem (Briegel, Raussendorf, 2001)

Measurement-based quantum computing on the grid graph states has the same computational power as quantum circuits.



Power of measurement-based quantum computing

Theorem (Nielsen, 2005)

For path graph states, measurement-based quantum computing is efficiently simulable on a classical computer.



Power of measurement-based quantum computing

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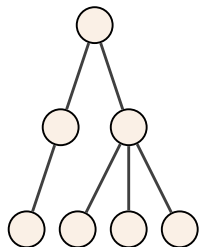
For path graph states, measurement-based quantum computing is efficiently simulable on a classical computer.



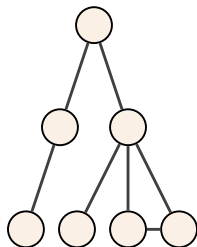
Intuition: no quantum power when the graph looks like a tree.

Tree-width

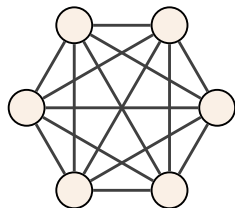
Informally, tree-width is a function in \mathbb{N} that measures how close a graph is to a tree.



tree-width = 1



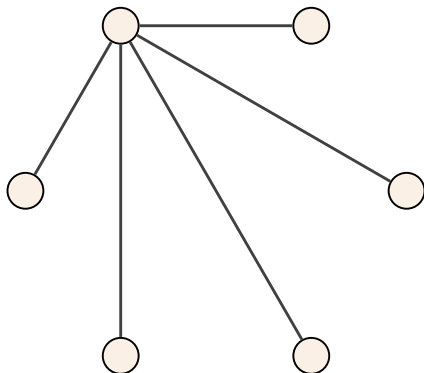
tree-width = 2



complete graph
on n vertices:
tree-width = $n - 1$

Tree-width does not pair well with local complementation

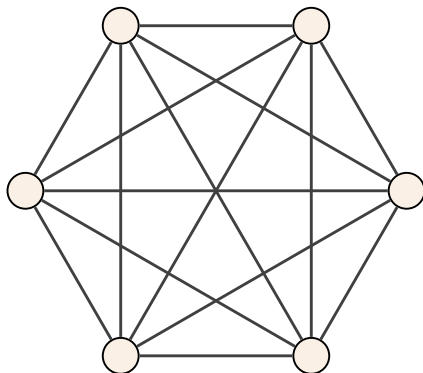
A star and a complete graph are the same up to local complementation.



tree-width = 1

Tree-width does not pair well with local complementation

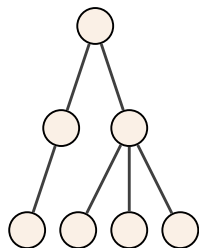
A star and a complete graph are the same up to local complementation.



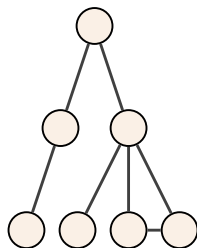
$$\text{tree-width} = n - 1$$

Rank-width

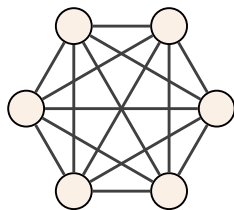
Informally, rank-width is like tree-width, but invariant by local complementation.



rank-width = 1



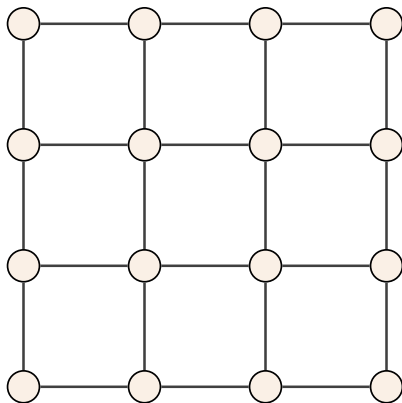
rank-width = 2



complete graph
on n vertices:
rank-width = 1

Rank-width of the square grid

The $m \times m$ square grid has rank-width $m - 1$ (Jelínek, 2008).



Low rank-width implies efficient classical simulation

Theorem (Van den Nest et al., 2006)

*If the **rank-width** of the graphs in a class \mathcal{G} grows at most **logarithmically** with the number of qubits, then measurement-based quantum computing on \mathcal{G} is efficiently simulable on a classical computer.*

Small rank-width = efficient classical simulation?

Small rank width implies efficient classical simulation.

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Is it an equivalence?

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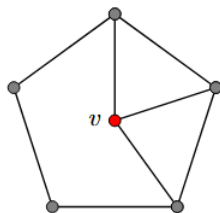
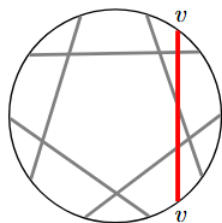
No!

Circle graphs

Circle graphs: definition

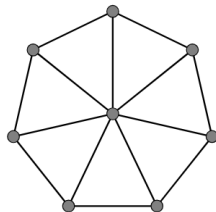
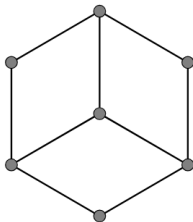
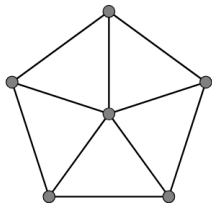
Definition

A circle graph is the intersection graph of a chord diagram.



Forbidden minors of circle graphs

Alternative definition: circle graphs are those graphs which do not have one of the 3 graphs below as a vertex-minor (Bouchet, 1994).



Importance of circle graphs

Circle graphs play in the theory of **vertex-minors** the role that **planar graphs**¹ play in the theory of **minors**².

¹planar = can be drawn in the plane without edges crossing.

²minor = obtained by vertex/edge deletion and edge contraction.

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Theorem (Grid theorem, Robertson and Seymour, 1986)

*For any **planar graph** G , there exists an integer r_G such that every graph with **tree-width** at least r_G has G as a **minor**.*

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Theorem (Grid theorem for vertex-minors, Geelen et al., 2020)

*For any **circle graph** G , there exists an integer r_G such that every graph with **rank-width** at least r_G has G as a **vertex-minor**.*

¹planar = can be drawn in the plane without edges crossing.

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Simulation of circle graphs

Proposition

Circle graphs have (in general) polynomial rank-width.

Simulation of circle graphs

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Circle graphs have (in general) polynomial rank-width.

However:

Theorem (Harrison et al., 2025)

Measurement-based quantum computing on circle graphs is efficiently simulable on a classical computer.

A conjecture that encompasses both examples

Conjecture (The simulation conjecture, Geelen)

*If we restrict ourselves to preparing graph states from any **proper vertex-minor-closed class** of graphs, then measurement-based quantum computing is efficiently simulable on a classical computer.*

→ Excluding a single graph as a vertex-minor forbids any quantum advantage.

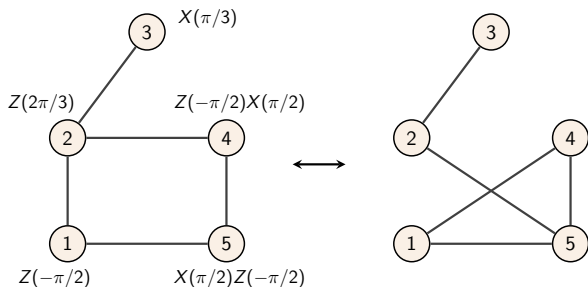
(Both graph states of bounded rank-width and circle graphs are proper vertex-minor closed classes.)

Classifying graph states

Local unitary equivalence

Definition

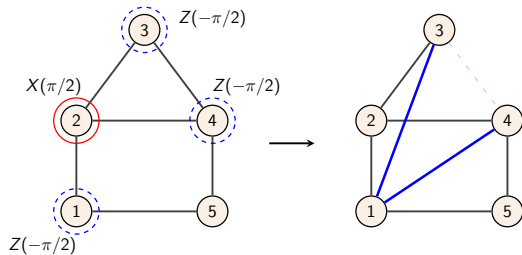
Two graph states are said local unitary equivalent if they are related by local quantum operations.



Local complementation and local unitary equivalence

Proposition

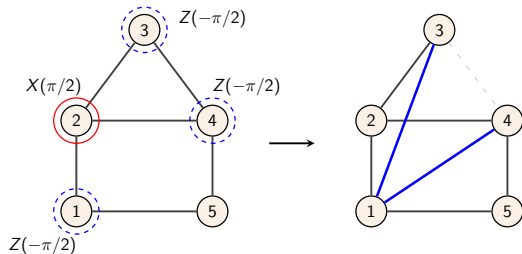
Related by local complementations \implies local unitary equivalent.



Local complementation and local unitary equivalence

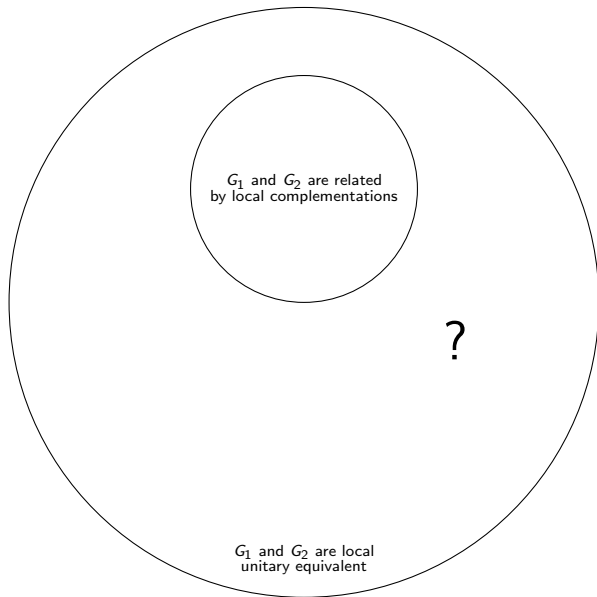
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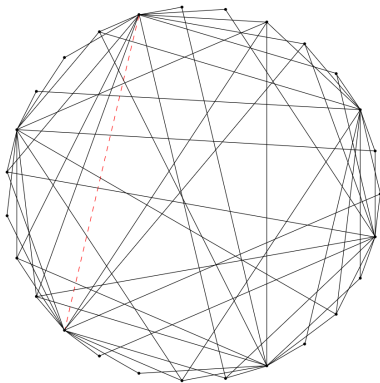
Is the converse true?

Hierarchy



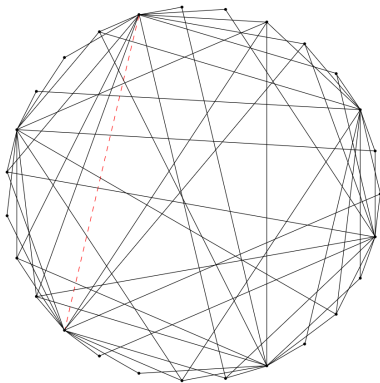
A counter-example

A pair of 27-qubit local unitary equivalent graph states, not related by local complementations (Ji et al. 2008):



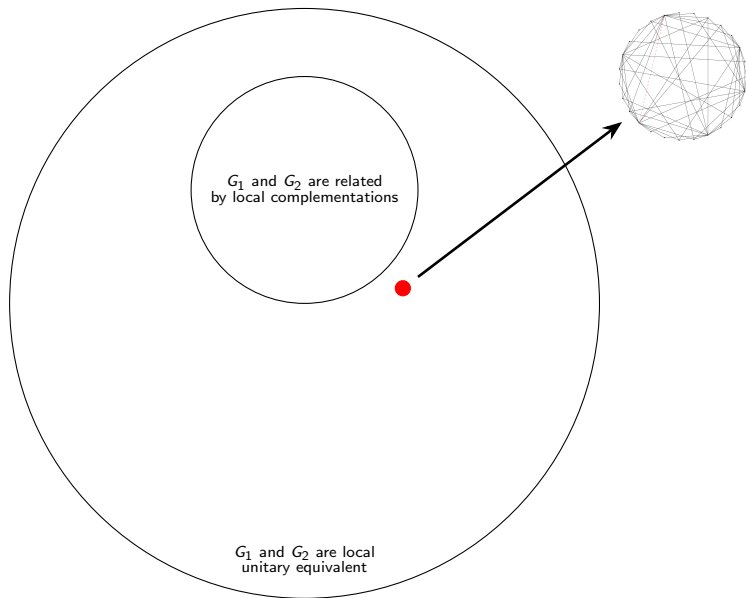
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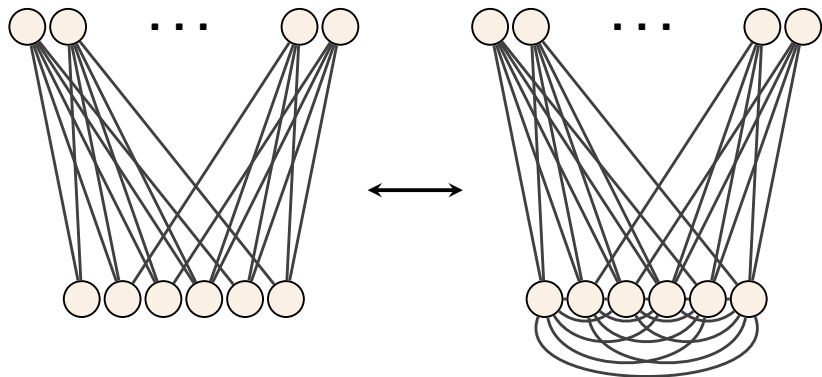
This counter-example is minimal (C. 2026). (one-month-old result!)

Hierarchy



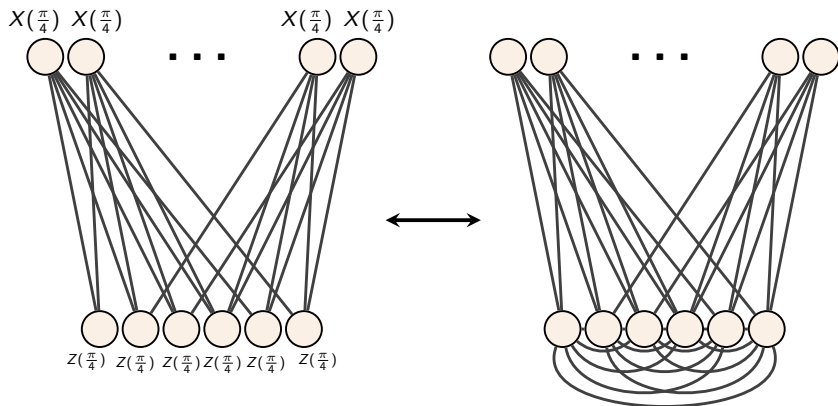
Another look at the 27-qubit counterexample

Equivalent counterexample (Tsimakuridze, Gühne, 2017).



Another look at the 27-qubit counterexample

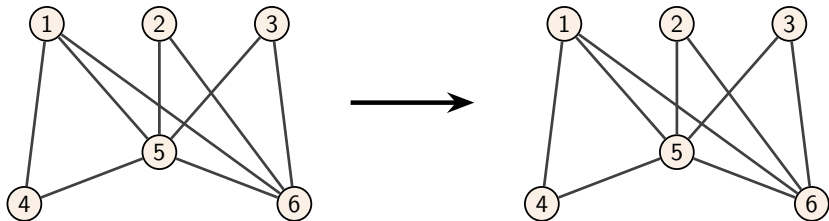
Equivalent counterexample (Tsimakuridze, Gühne, 2017).



Generalizing local complementation to capture
local unitary equivalence

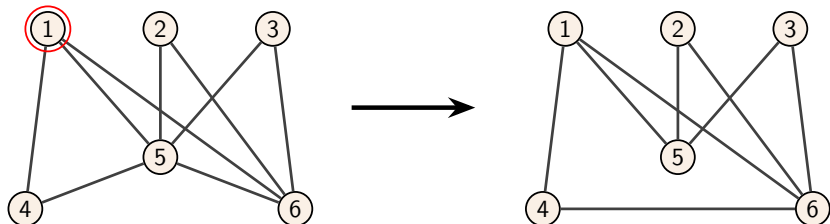
A refinement of idempotent local complementations

A sequence of local complementations may leave the graph invariant.



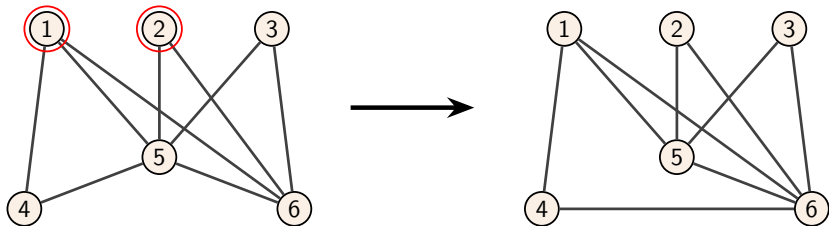
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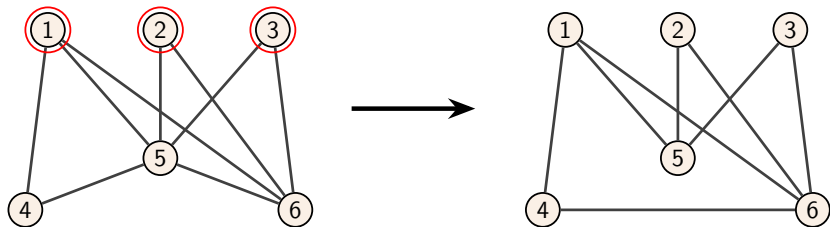
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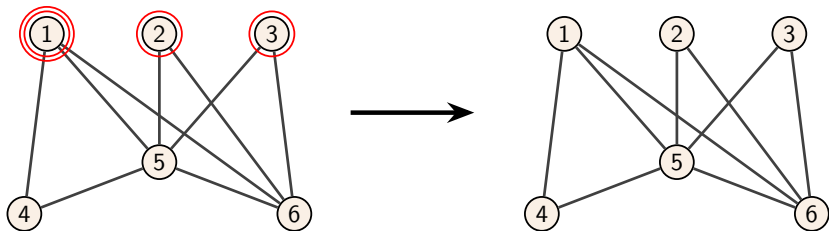
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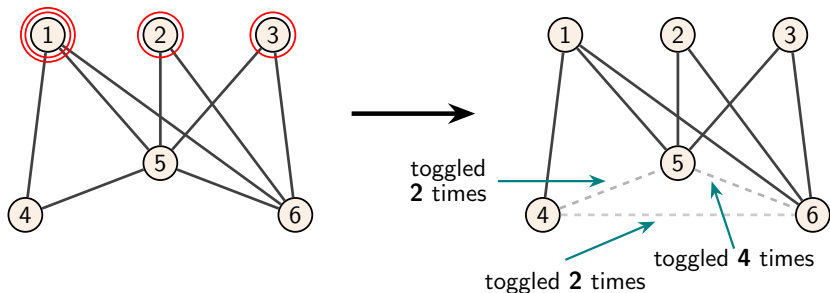
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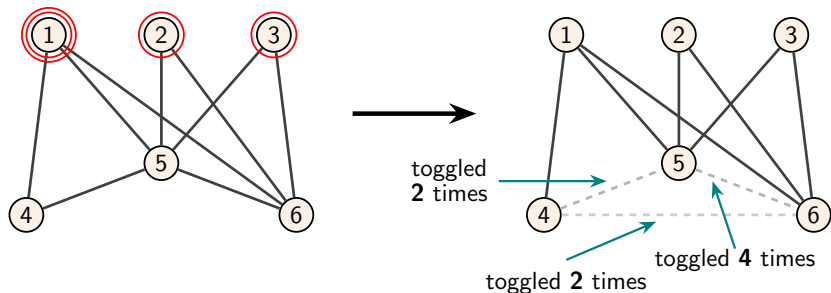
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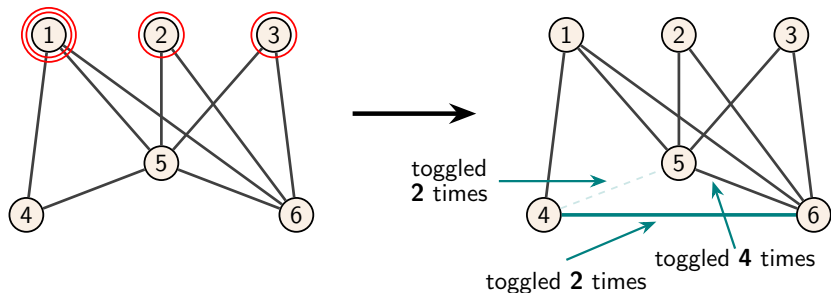


A **2-local complementation** consists in toggling every edge that was toggled 2 mod 4 times by the idempotent local complementations.

+ additional conditions on the edges

A refinement of idempotent local complementations

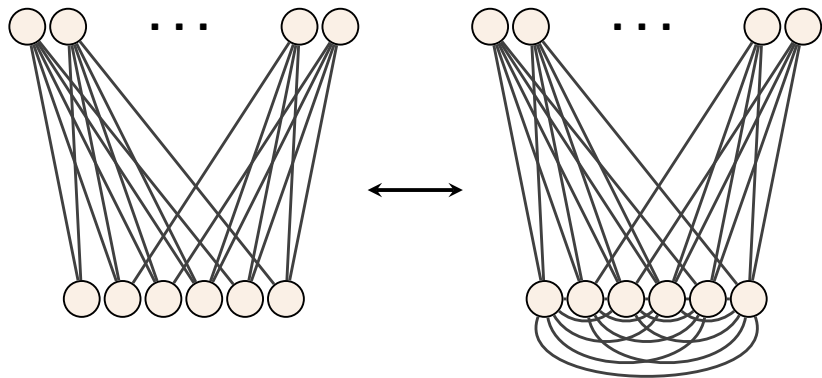
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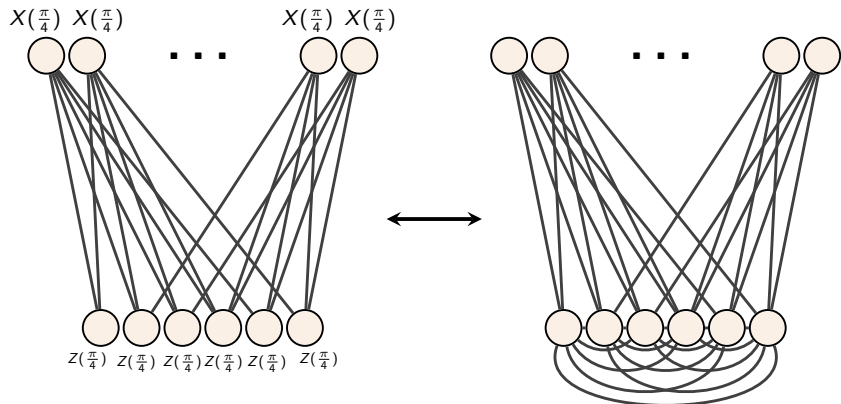
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Example of a 2-local complementation



Example of a 2-local complementation



3-local complementation is a refinement of idempotent 2-local complementation, and so on...

→ Infinite family of graphical operations parametrised by an integer r :

r -local complementations

1-local complementation = local complementation.

r -local complementation captures local unitary equivalence

Theorem (C, Perdrix, 2025)

Two graph states are local unitary equivalent if and only if the two corresponding graphs are related by r -local complementations where

$$r = O(\log(n))$$

r -local complementation captures local unitary equivalence

Theorem (C, Perdrix, 2025)

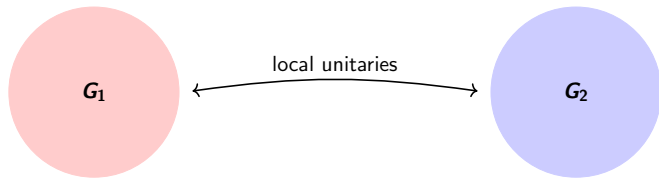
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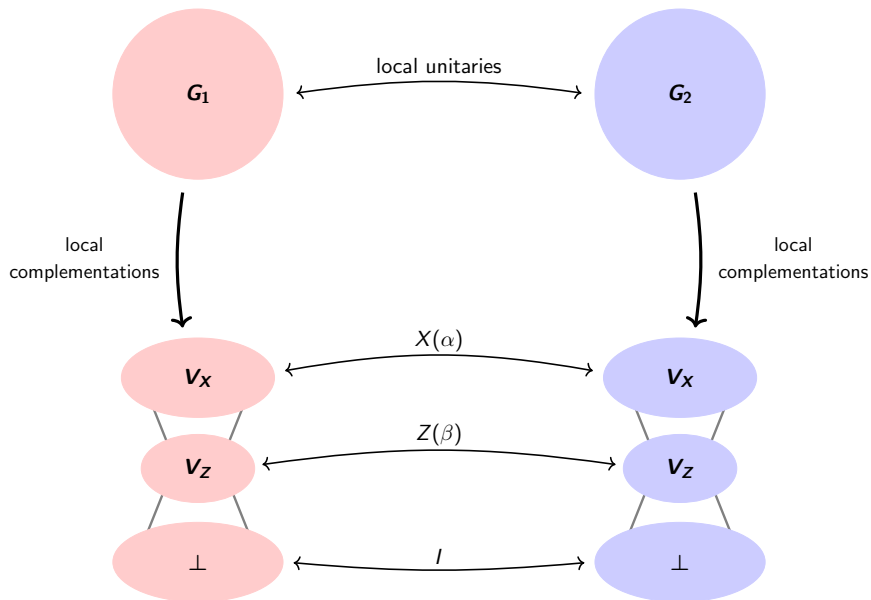
Theorem (C, Perdrix, 2025)

There exists an algorithm that decides if two graph states are local unitary equivalent with runtime $n^{\log_2(n)+O(1)}$.

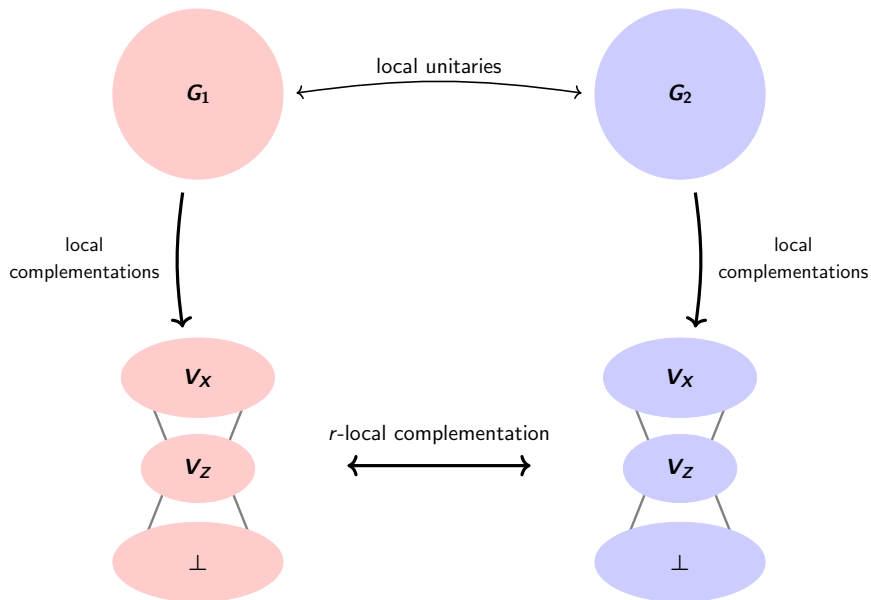
Proof sketch: Standard form for graph states



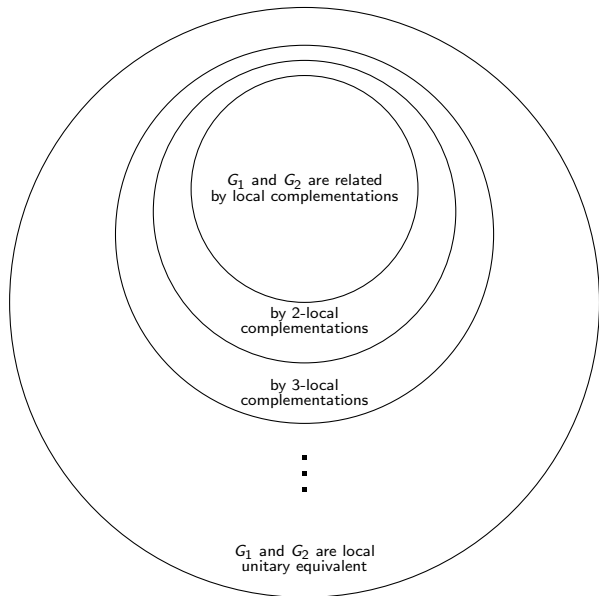
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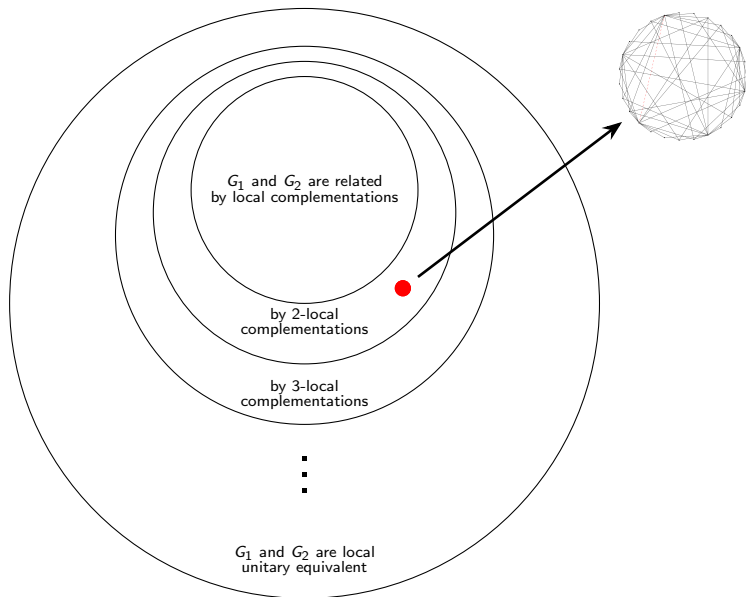
Proof sketch: Standard form for graph states



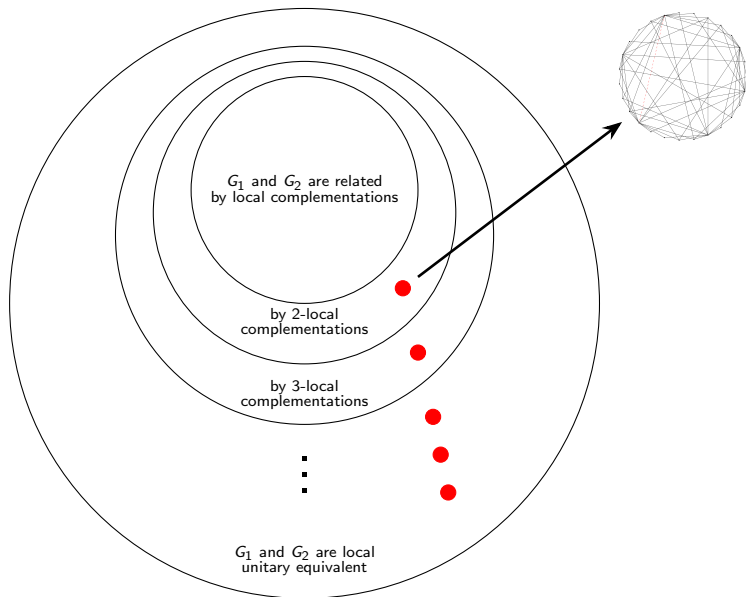
An infinite hierarchy of local equivalences



An infinite hierarchy of local equivalences



An infinite hierarchy of local equivalences



Some open questions

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- Polynomial-time algorithm for local unitary equivalence? Is it in NP?

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- Simulation conjecture?
- Something like vertex-minors but for all local quantum operations?

Thank you!

