Small k-pairable states

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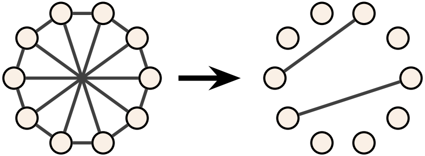
Quantum days at AMU - 20/09/23
arXiv:2309.09956



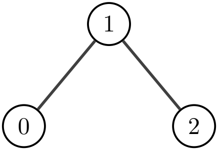
Graph states

Definition (Graph state)

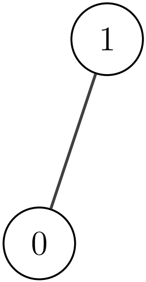
Given a graph $G = (V, E)$, a graph state $|G\rangle$ is a quantum state written as :

$$|G\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in 2^V} (-1)^{|G[x]|} |x\rangle$$

where $|G[x]|$ is the number of edges in the subgraph induced by x.



$$|G\rangle = \frac{1}{\sqrt{8}} (|000\rangle + |001\rangle + |010\rangle - |011\rangle + |100\rangle + |101\rangle - |110\rangle + |111\rangle)$$



$\sim$ EPR-pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

k-pairability

Notion introduced by Sergey Bravyi, Yash Sharma, Mario Szegedy, Ronald de Wolf in “Generating k EPR-pairs from an n-party resource state” (2022).
Motivation : Quantum communication networks.

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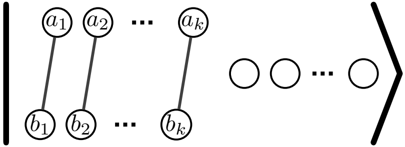
Definition (k-pairable state)

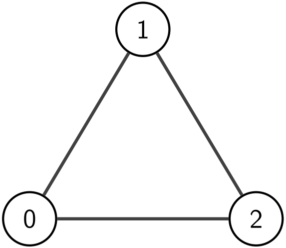
A n-qubit quantum state $|\psi\rangle$ is k-pairable if for any k disjoint pairs of qubits $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_k, b_k\}$ there exits an LOCC (Local Operations and Classical Communication) protocol that transforms $|\psi\rangle$ into $|\pi\rangle$.

$|\pi\rangle =$ 

$\sim$ k EPR-pairs and $n-2k$ isolated qubits

k-pairable states : example 1

$|K_3\rangle$ is 1-pairable.



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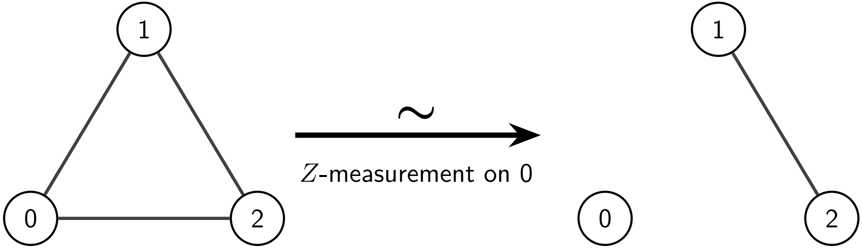

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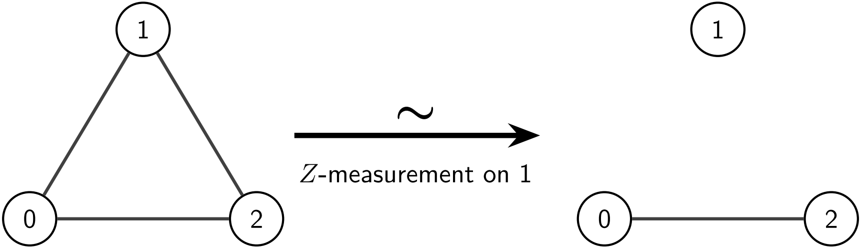

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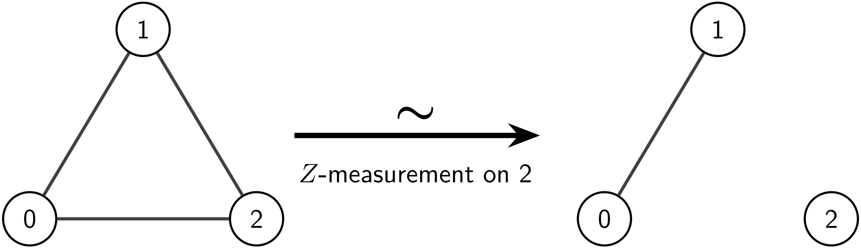
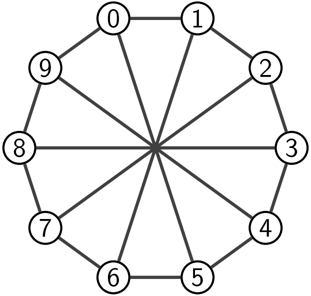


k-pairable states : example 2

$|G\rangle$ is 2-pairable.



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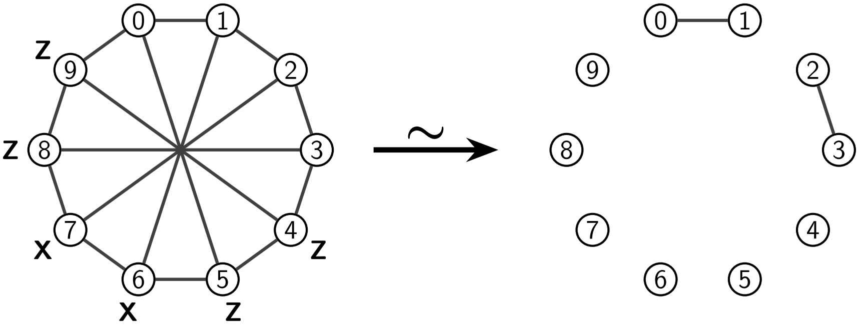

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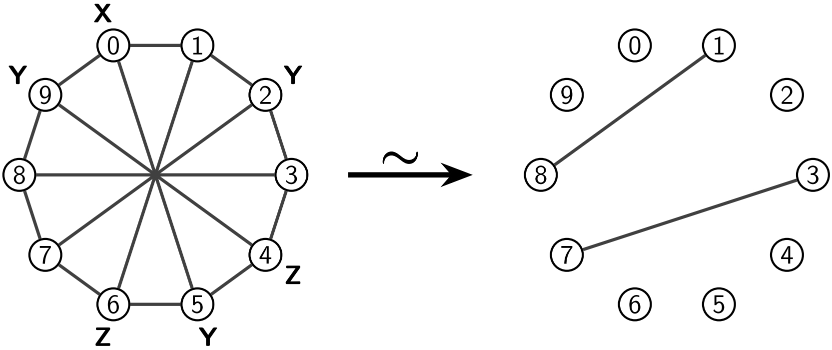
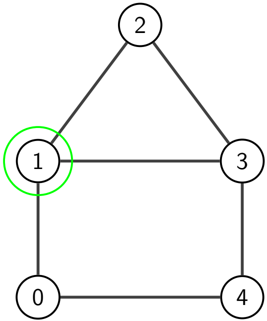


Sufficient condition on G for $|G\rangle$ to be k-pairable ?

Definition (Local complementation)

Given a graph G, a local complementation on a vertex u consists in complementing the (open) neighborhood of u in G.



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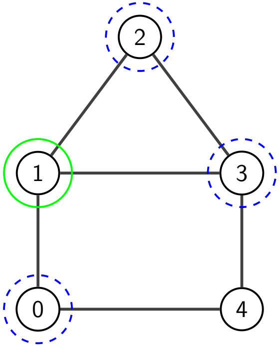

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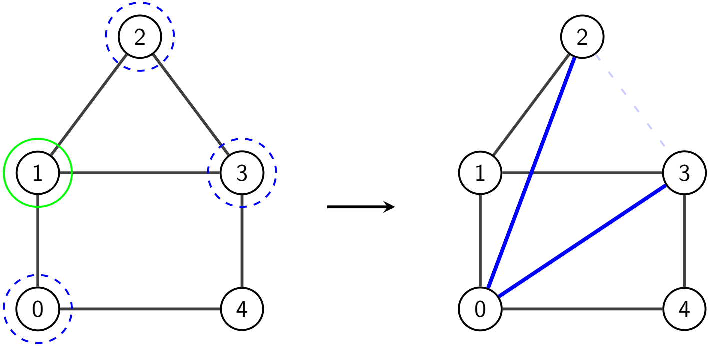

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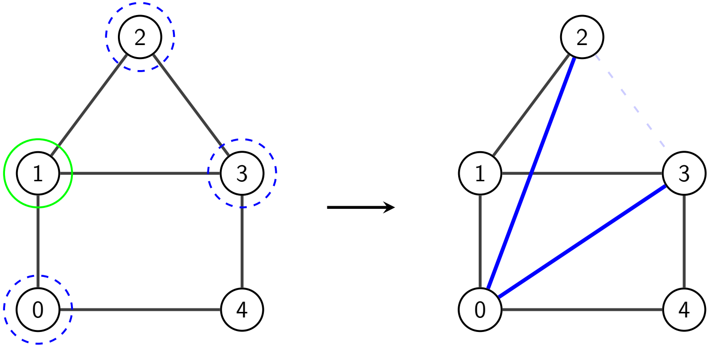
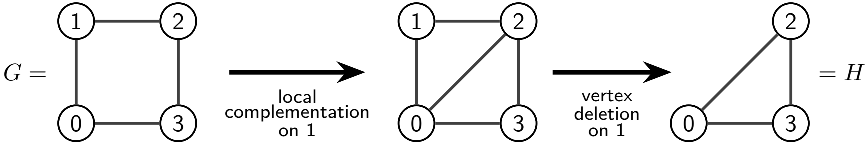


A local complementation on G can be implemented by local operations on $|G\rangle$.

Vertex-minors

Definition (Vertex-minor)

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ such that $V_H \subseteq V_G$, H is a vertex-minor of G if H can be obtained from G by means of local complementations and vertex deletions.



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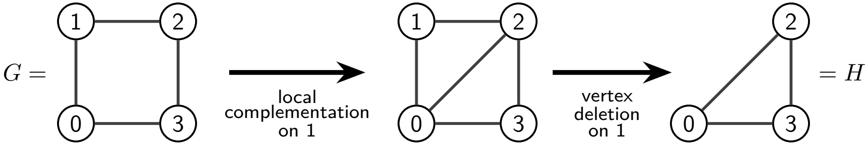
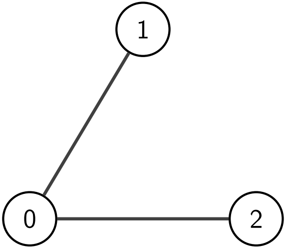


Proposition

A graph state $|G\rangle$ is k-pairable if any perfect matching on any $2k$ vertices is a vertex-minor of G.

k-pairable states : example 1

$|G\rangle$ is 1-pairable.



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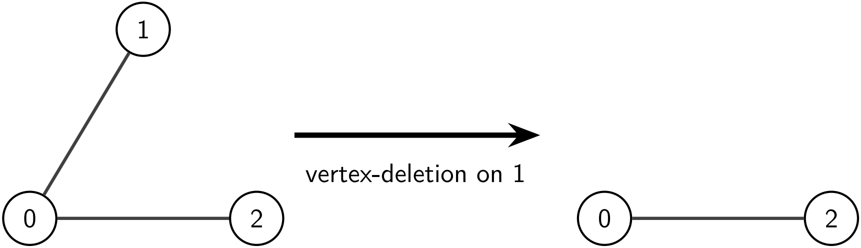

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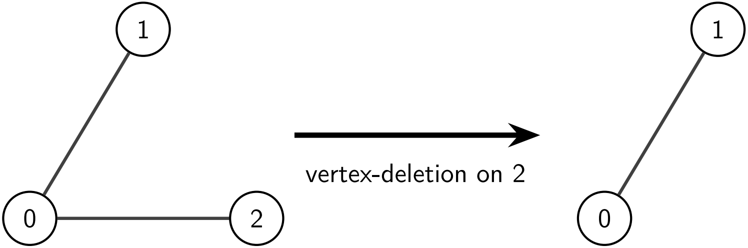

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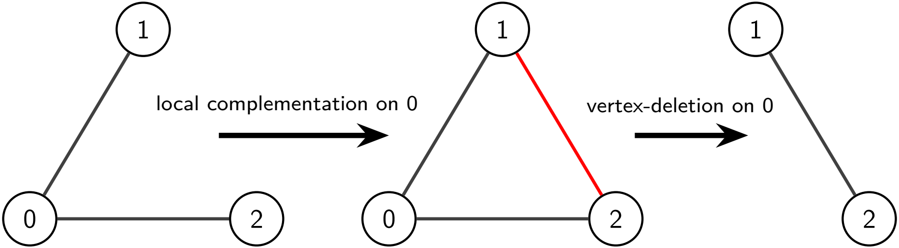
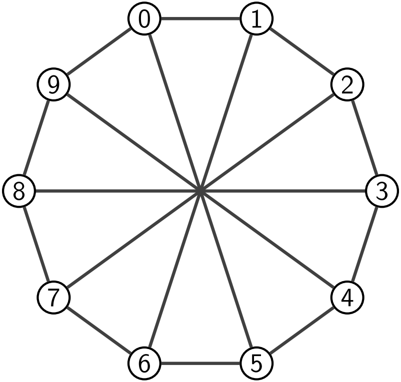


k-pairable states : example 2

$|G\rangle$ is 2-pairable.



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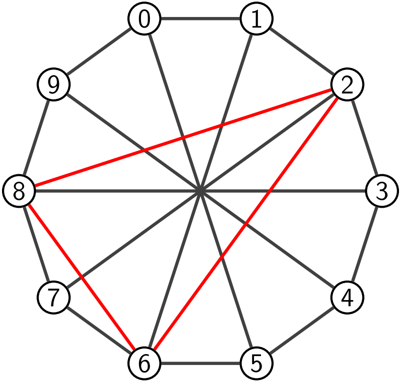

To induce the pairs $(0, 1)$ and $(2, 3)$: local complementation on 7

k-pairable states : example 2

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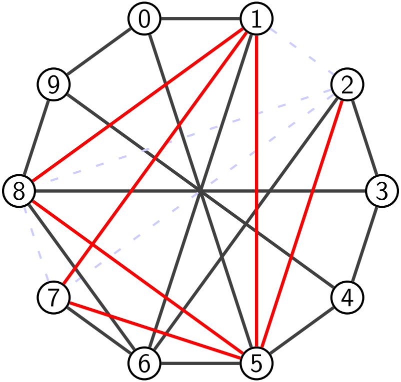

To induce the pairs $(0, 1)$ and $(2, 3)$: local complementation on 7, then local complementation on 6

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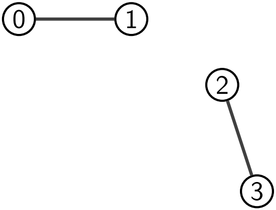

To induce the pairs $(0, 1)$ and $(2, 3)$: local complementation on 7, then local complementation on 6, then vertex-deletions on 4,5,6,7,8 and 9.

To sum up

- A quantum state is said k-pairable if we can transform it into any k EPR-pairs on any $2k$ qubits using LOCC protocols.
- We have a condition on a graph G for the corresponding graph state $|G\rangle$ to be k-pairable. This condition is useful for the proofs of our results.

Towards small k-pairable states

Proposition (Bravyi et al. 2022)

For any k, there exists a k-pairable state on $n = 2^{3k}$ qubits.

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Does there exists k-pairable state where $n = poly(k)$?
Yes !

Proposition

For any k, there exists a k-pairable state on $n = O(k^3 \ln^3(k))$ qubits.

k-vertex-minor-universal graphs

Recall that a sufficient condition on $|G\rangle$ to be k-pairable is that G has every perfect matching on $2k$ vertices as its vertex-minors.

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Natural extension :

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A graph G is *k-vertex-minor-universal* if any graph on any k vertices is a vertex-minor of G.

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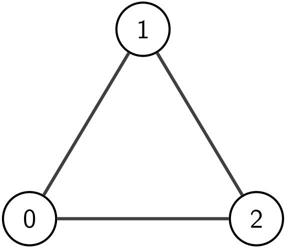
Corollary

If G is a $2k$-vertex-minor-universal graph, then $|G\rangle$ is a k-pairable graph state.

k-vertex-minors-universal graphs : example 1

K_3 is 2-vertex-minor-universal.



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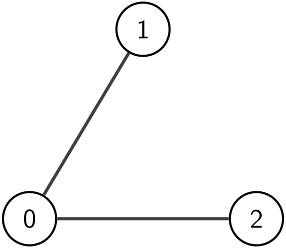
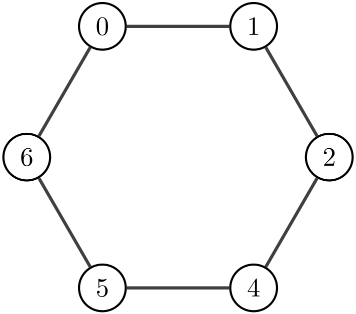


k-vertex-minors-universal graphs : example 2

C_6 is 3-vertex-minor-universal.



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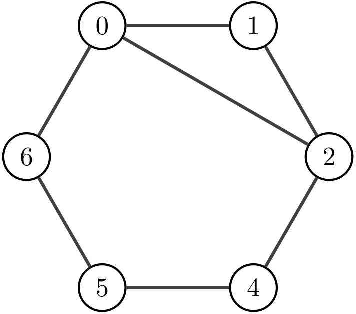

Towards small k-vertex-minors-universal graphs

Proposition

For any k, there exists a k-vertex-minor-universal graph of order $n = O(k^4 \ln(k))$.

Some other results

- Upper bounds : a graph state $|G\rangle$ is not $\left(\left\lceil \frac{\delta_{\mathsf{loc}}(G)}{2} \right\rceil + 1\right)$-pairable, and is not $\left(\left\lceil \frac{\tau(G)+\log_2(\tau(G))}{4} \right\rceil + 1\right)$-pairable.

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- Alternative version of k-pairability that is robust in the presence of errors or malicious parties
- Practical results :

graph G	order	$\delta_{\text{loc}}(G)$	1-p?	2-vmu?	3-vmu?	2-p?	4-vmu?	5-vmu?	3-p?
K_2	2	1	yes	no					
K_3	3	2	yes	yes	no				
C_6	6	2	yes	yes	yes	no	no	no	no
"Wheel" graph	10	3	yes	yes	yes	yes	yes	no	no
Petersen graph	10	3	yes	yes	yes	yes	yes	no	no
13-Paley graph	13	4	yes	yes	yes	yes	yes	no	no
17-Paley graph	17	4	yes	yes	yes	yes	yes	yes	no
29-Paley graph	29	10	yes	yes	yes	yes	yes	yes	yes

Conclusion

Results on k-pairable states and k-vertex-minor-universal graphs.

Future work:

- Explicit constructions
- k-pairability $= 2k$-vertex-minor-universality ?

Thanks

arXiv:2309.09956